Dynamical aperture beyond perturbations: strong localization approach

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APM'2011, July , 2011

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Abstract

We present the applications of the multiresolution analysis approach in the constrained variational framework for calculation of dynamical aperture for particle/beam motion in accelerators. We construct an exact multiscale representation by decomposition via nonlinear high-localized eigenmodes, which allows to control contributions to motion from the whole underlying hidden multiscale structure. We consider a qualitative approach to the aperture problem based on the detailed analysis of smoothness classes in the underlying functional space.

Introduction

The estimation of the dynamic aperture of accelerators is an important, complicated and long standing problem. From the formal point of view the aperture is some border between two types of dynamics: relative regular and predictable motion along of acceptable orbits or fluxes of orbits corresponding to KAM tori and stochastic motion with particle losses blown away by the Arnold diffusion and/or chaotic motions. According to the standard point of view this transition is being done by some analogues with map technique [1]. Consideration for aperture of n-pole Hamiltonians with kicks

$$H = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2 + \frac{1}{3!B\rho}\frac{\partial^2 B_z}{\partial x^2}(x^3 - 3xy^2)L\sum_{k=-\infty}^{\infty}\delta(s - kL) + \dots \tag{1}$$

is done by linearisation and discretization of canonical transformation and the result resembles (pure formally) standard mapping.

This leads, by using the Chirikov criterion of resonance overlapping, to evaluation of aperture via amplitude of the following global harmonic representation ([1] for designations and description):

$$x^{(n)}(s) = \sqrt{2J_{(n)}\beta_x(s)} \cdot \cos\left(\psi_1 - \frac{2\pi\nu}{L}s + \int_0^s \frac{\mathrm{d}s'}{\beta_x(s')}\right). \qquad (2)$$

The goal of this paper is is two-fold. In part 2, we consider some qualitative criterion which is based on the attempts of more realistic understanding of the observable existing difference between motion in KAM regions and stochastic regions: motion in KAM regions may be described by regular functions only (without the influence of complicated internal structures leading to nonuniform hyperbolicity generating chaos) while motion in stochastic regions/layers may be described by functions with internal self-similar structures (definitely, created by actions of symmetry generated groups, like discrete groups, or by actions of hidden symmetries of background functional space, like affine group in the most simple case) i.e. fractal type functions as realizations of proper orbits.

So, the difference is fundamental because it is related to different functional spaces in background and different hidden structures inside them. The wavelet analysis (multiresolution technique) approach [2], [3] provides the possibilities for relative analytical description based on calculations of wavelet coefficients/wavelet transform asymptotics.
In part 3, we consider an attempt to the same problem on the more quantitative level. In the constrained variational framework we construct an explicit representation for all dynamical variables as expansions in the base of (nonlinear) periodic high-localized eigenmodes. Such the approach provides, in principle, the possibility for the control of aperture behaviour in the space of machine parameters.

Qualitative analysis

The fractal or chaotic image is a function (distribution) which has structure at all underlying scales. Such objects have additional nontrivial details on any level of resolution. But they cannot be represented by smooth functions, because they resemble constants at small scales [2], [3]. We need to find self-similarity behaviour during movement to small scales for the functions describing non-regular motion. So, if we look on a "fractal" function f (e.g. the Weierstrass function) near an arbitrary point at different scales, we find the same function up to the scaling factor. Consider the fluctuations of such function f near some point x_0

$$f_{loc}(x) = f(x_0 + x) - f(x_0), \tag{3}$$

then we have the renormalization (group)–like behaviour/transformation

$$f_{x_0}(\lambda x) \sim \lambda^{\alpha(x_0)} f_{x_0}(x), \tag{4}$$

where $\alpha(x_0)$ is the so-called local scaling exponent or Hölder exponent of the function f at x_0. According to [3] general functional spaces and scales of spaces can be characterized through wavelet coefficients or wavelet transforms.

Let us consider continuous wavelet transform

$$W_g f(b,a) = \int_{R^n} \mathrm{d}x \frac{1}{a^n} \bar{g}\left(\frac{x-b}{a}\right) f(x),$$

$b \in R^n$, $a > 0$, w.r.t. analyzing wavelet g, which is strictly admissible, i.e.

$$C_{g,g} = \int_0^\infty \frac{\mathrm{d}a}{a} |\hat{g}(\bar{a}\ k)|^2 < \infty.$$

Wavelet transform has the following covariance property under action of the underlying affine group:

$$W_g(\lambda a, x_0 + \lambda b) \sim \lambda^{\alpha(x_0)} W_g(a, x_0 + b). \qquad (5)$$

So, if the Hölder exponent of (distribution) $f(x)$ around the point $x = x_0$ is $h(x_0) \in (n, n+1)$, then we have the following behaviour of $f(x)$ around $x = x_0$:

$$f(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n + c|x - x_0|^{h(x_0)}.$$

Let the analyzing wavelet has n_1 $(> n)$ vanishing moments, then

$$W_g(f)(x_0, a) = Ca^{h(x_0)} W_g(f)(x_0, a) \tag{6}$$

and

$$W_g(f)(x_0, a) \sim a^{h(x_0)},$$

when $a \to 0$. But if $f \in C^\infty$ at least in the point x_0, then

$$W_g(f)(x_0, a) \sim a^{n_1},$$

when $a \to 0$. This shows that the localization of wavelet coefficients at small scale is linked to local regularity. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular. So, transition from regular motion to chaotic one may be characterised as the changing of the Hölder/scaling exponent of the function which describes motion. This gives a criterion of the appearance of fractal behaviour and may determine, at least in principle, the dynamic aperture.

Constrained problem for quasi-periodic orbits

We consider the extension of our approach [4]–[15] to the case of constrained quasi-periodic trajectories. The equations of motion corresponding to (polynomial) Hamiltonian families like (1) may be formulated as a particular case of the general system of ordinary differential equations

$$dx_i/ds = f_i(x_j, s), \quad (i, j = 1, ..., 2n),$$

where f_i are not more than rational functions of dynamical variables x_j and have arbitrary dependence on time but with periodic boundary conditions. Let us consider our set-up as an operator equation for the operator S which satisfies the equation

$$S(H, x, \partial/\partial s, \partial/\partial x, s) = 0, \tag{7}$$

which is polynomial/rational in $x = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ and have arbitrary dependence on s together with the additional operator $C(H, x, \partial/\partial t, \partial/\partial x, s)$, which is an operator describing some constraints, as differential as integral, on the set of dynamical variables. E.g., we may fix a part of non-destroying integrals of motion (e.g., energy) or areas in phase space (fluxes of orbits).

So, we may consider our problem as the problem of constructing orbits described by Hamiltonian families like (1), but with needful geometrical-like properties. In this way we may fix a given acceptable aperture or vice versa by feedback via parametrisation of orbits by coefficients of the initial dynamical problem we may control different levels of aperture as a function of the parameters of the system (7) under consideration. As a result, our variational problem is formulated for pair of operators (C, S) on the extended set of dynamical variables, which includes Lagrangian multipliers λ. Then we use the (weak) variation formulation

$$\int < (S + \lambda C)x, y > \mathrm{d}t = 0. \tag{8}$$

We start with a hierarchical sequence of approximations spaces:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots , \tag{9}$$

and the corresponding expansions:

$$x^N(s) = \sum_{r=1}^{N} a_r \psi_r(s), \qquad y^N(s) = \sum_{k=1}^{N} b_k \psi_k(s). \tag{10}$$

As a result of our approach [4]–[15], we obtain, from (7), the following reduced system of algebraical equations (RSAE) on a set of unknown coefficients a_i of expansions (10), the so-called Generalized Dispersion Relations:

$$L(S_{ij}, C_{kl}, a, \Lambda) = 0, \tag{11}$$

where operator L is algebraization of the initial problem (7) after variational procedure. In addition, in the general situation, we need to find also objects Λ

$$\Lambda^{d_1 d_2 \dots d_n}_{\ell_1 \ell_2 \dots \ell_n} = \int\limits_{-\infty}^{\infty} \prod \psi^{d_i}_{\ell_i}(x) \mathrm{d}x. \tag{12}$$

We consider the procedure of their calculations in the case of quasi/periodic boundary conditions in the bases of periodic wavelet functions with periods T_i on the interval [0, T] for the corresponding expansions (10) inside our variational-wavelet approach [4]–[15].

Periodization procedure gives

$$\hat{\varphi}_{j,k}(x) \equiv \sum_{\ell\in Z} \varphi_{j,k}(x-\ell), \qquad (13)$$

$$\hat{\psi}_{j,k}(x) \equiv \sum_{\ell\in Z} \psi_{j,k}(x-\ell).$$

So, $\hat{\varphi}, \hat{\psi}$ are periodic functions on the interval [0, T]. Because $\varphi_{j,k} = \varphi_{j,k'}$ if $k = k' \mathrm{mod}(2^j)$, we may consider only $0 \leq k \leq 2^j$ and as consequence, our generic multiresolution decomposition for the background functional space has the form [16]:

$$\bigcup_{j\geq 0} \hat{V}_j = L^2[0, T]$$

with

$$\hat{V}_j = \mathrm{span}\{\hat{\varphi}_{j,k}\}_{k=0}^{2j-1}.$$

Integration by parts and periodicity give useful relations between the objects (12) in the particular quadratic case $(d = d_1 + d_2)$:

$$\Lambda^{d_1,d_2}_{k_1,k_2} = (-1)^{d_1} \Lambda^{0,d_2+d_1}_{k_1,k_2}, \qquad (14)$$

$$\Lambda^{0,d}_{k_1,k_2} = \Lambda^{0,d}_{0,k_2-k_1} \equiv \Lambda^{d}_{k_2-k_1}.$$

So, any 2-tuple can be represented by Λ_k^d. Then our second (after (11)) additional algebraic (linear) problem is reduced according to [16] to the eigenvalue problem for $\{\Lambda_k^d\}_{0\le k\le 2^j}$ by creating a system of 2^j homogeneous relations in Λ_k^d and additional inhomogeneous equations. If we consider generic dilation equation in the form

$$\varphi(x)=\sqrt{2}\sum_{k\in Z}h_k\varphi(2x-k),$$

then we have the following homogeneous relations

$$\Lambda_k^d=2^d\sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1}h_m h_\ell\Lambda_{\ell+2k-m}^d, \qquad (15)$$

or in such a clear form

$$A\lambda^d=2^d\lambda^d, \qquad \text{where} \qquad \lambda^d=\{\Lambda_k^d\}_{0\le k\le 2^j}.$$

Inhomogeneous equations are:

$$\sum_\ell M_\ell^d\Lambda_\ell^d=d!2^{-j/2}, \qquad (16)$$

where objects $M_\ell^d(|\ell|\le N-2)$ can be computed by recursive procedure

$$M_\ell^d=2^{-j(2d+1)/2}\tilde{M}_\ell^d, \qquad (17)$$

$$\tilde{M}_\ell^k=<x^k,\varphi_{0,\ell}>=\sum_{j=0}^{k}\binom{k}{j}n^{k-j}M_0^j, \quad \tilde{M}_0^\ell=1.$$

Anyway, all these problems are the standard linear algebraical problems solvable by means of standard routines and it is not a big deal to solve explicitly all that to provide the base for numerical modeling and simulation. Then, we may solve generic Generalized Dispersion Relations, RSAE (11) (what is much more complicated procedure), and determine unknown coefficients of the formal expansion (10) corresponding to multiresolution decomposition of the background functional space. Therefore, in such a manner, we obtain the solution of our initial problem. It should be noted that if we consider only truncated expansion with N terms then we have, from (11), a system of $N \times 2n$ algebraical equations and the degree of this algebraical system coincides with the degree of the initial differential system. As a result, we obtained the following explicit representation for (quasi)periodic orbits/trajectories in the base of periodized (period T_i) localized wavelets (10) (properly chosen accordingly to the background functional space):

$$x_i(s) = x_i(0) + \sum_k a_i^k \psi_k^i(s), \quad x_i(0) = x_i(T_i). \tag{18}$$

Because the affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contributions to the final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to the level j of resolution, or to the scale j. The final multiscale/multiresolution representation for solution has the following form

$$x(s) = x_N^{slow}(s) + \sum_{j \geq N} x_j(\omega_j s), \quad \omega_j \sim 2^j, \tag{19}$$

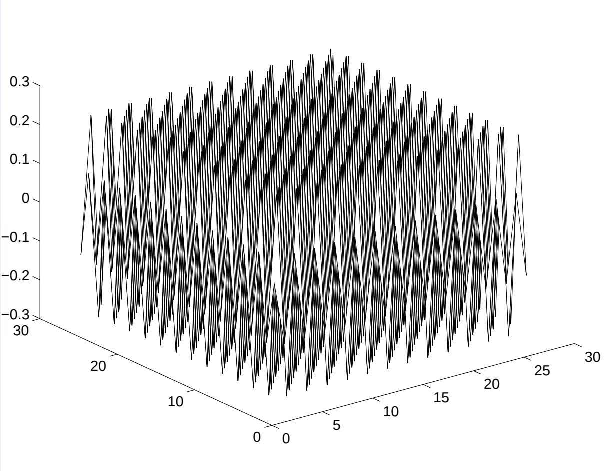

Figure: Periodic structure on level 6 of multiresolution.

which corresponds to the full multiresolution expansion in all time scales.

Formula (19) gives us expansion into the slow part x_N^{slow} and fast oscillating parts (generating out-of-KAM behaviour) for an arbitrary N. So, we may move from coarser scales of resolution to a finer scale for obtaining more detailed information about our dynamical process. The first term in the RHS of the equation (19) corresponds on the global level of the function space decomposition to resolution space and the second one to detail space. In this way we give contributions to our full solution from each scale of resolution or each time scale. On Fig. 1 we present (quasi) periodic regime on the section $x - p_x$ corresponding to the model (1).

So, in such a way, by means of multiresolution decomposition for the explicit solution of the properly/geometrically constrained Hamiltonian set-up we provide analytical background for the dynamical aperture evaluation. Definitely, above we used nothing related to the orthodox perturbation technique.