Thorny Path to Fusion in Plasma: Confinement State As a Waveleton

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Abstract

A fast and efficient numerical-analytical approach is proposed for description of complex behaviour in non-equilibrium ensembles in the BBGKY framework. We construct the multiscale representation for hierarchy of partition functions by means of the variational approach and multiresolution decomposition. Numerical modeling shows the creation of various internal structures from fundamental localized (eigen)modes. These patterns determine the behaviour of plasma. The localized pattern (waveleton) is a model for energy confinement state (fusion) in plasma.

"A magnetically confined plasma cannot be in thermodinamical equilibrium"
Unknown author ... Folklore

1 Introduction.

It is well known that fusion problem in plasma physics could be solved neither experimentally nor theoretically during last fifty years. At the same time, during this long period other areas of physics and engineering demonstrated vast growth, on the level of both theoretical understanding and practical smart realizability. Because financing contributions in this area definitely exceeds that of almost all other areas of Physics, it seems that there are the serious obstacles which prevent real progress in the problem of real fusion as the main subject in the area [1,2]. Of course, it may be a result of some unknown no-go theorem(s) but it seems that the current theoretical level definitely demonstrates that not all possibilities, at least on the level of theoretical and matematical modeling, are exhausted. Surely, it is more than clear that perturbations, linearization, PIC or MC do not exhaust all instruments which we have at our hands on the route to theoretical understanding and predictions. Definitely, we need much more to have influence on almost free-of-theoretical-background work of experimentalists and engineers who contribute to ITER, NIF and other related top level projects. So, this paper and related one [3] can be considered as a small contribution to an attempt to avoid the existing obstacles appeared on the main roads of current plasma physics, especially along thorny path to solution of fusion problem. Definitely, the first thing which we need to change is a framework of generic mathematical methods which can help to improve the current state of the theory. Our postulates (conjectures) are as follows [4], [7]–[23]:

A) The fusion problem (at least at the first step) must be considered as a problem inside the (non) equilibrium ensemble in the full phase space. It means, at least, that:

A1) our dynamical variables are partitions (partition functions, hierarchy of N-points partition functions),

A2) it is impossible to fix a priori the concrete distribution function and postulate it (e.g. Maxwell-like or other concrete (gaussian-like or even not) distributions) but, on the contrary, the proper distrubution(s) must be the solutions of proper (stochastic) dynamical problem(s), e.g., it may be the well-known framework of BBGKY hierarchy of kinetic equations or something similar. So, the full set of dynamical variables must include partitions also.

B) Fusion state = (meta) stable state (with minimum entropy and zero measure) in the space of partitions on the whole phase space in which most of energy of the system is concentrated in the relatively small area (preferable with measure zero) of the whole domain of definition in the phase space during the time period which is enough to take reasonable part of it outside for possible usage. From the formal/mathematical point of view it means that:

B1) fusion state must be localized (first of all, in the phase space),

B2) we need a set of building blocks, localized basic states or eigenmodes which can provide

B3) the creation of localized pattern which can be considered as a possible model for plasma in a fusion state. Such pattern must be:

B4) (meta) stable and controllable, because of obvious reasons. So, the main courses are:

C1) to present smart localized building blocks which may be not only useful from point of view of analytical statements, such as the best possible localization, fast convergence, sparse operators representation, etc, but also exist as real physical fundamental modes,

C2) to construct various possible patterns with special attention to localized pattern which can be considered as a needful thing in analysis of fusion;

C3) after points C1 and C2 in ensemble (BBGKY) framework to consider some standard reductions to Vlasov-like and RMS-like equations (following the set-up from well-known results [2]) which may be useful also. These particular cases may be important as from physical point of view as some illustration of general consideration [3].

The lines above are motivated by our attempts to analyze the hidden internal contents of the phrase mentioned in the epigraph of this paper: "A magnetically confined plasma cannot be in thermodinamical equilibrium." Also, it should be noted that our results below can be applied to any scenario (fusion, ignition, etc): we describe pattern formation in arbitrary non-equilibrium ensembles.

2 Motivations

It is obvious that any reasonable set-up for analysis of fusion leads to very complex system and one hardly believes that such system can be analyzed by means of almost exhausted methods like perturbations, linearization, etc. At the same time, because such complex chaotic/stochastic dynamics is overcompleted by short- and long-living fluctuations, instabilities, etc one needs to find something more proper than usual plane waves or gaussians for modeling a complicated complex behaviour. For reminiscences we may consider simple standard soliton equations like KdV, KP or sine-Gordon ones. It is well-known that neither linearization, nor perturbations, nor plane-wave-like approximations are proper for the reasonable analysis of such equations in contrary to wave or other simple linear equations: it is impossible to approximate the spectrum of such models (solitons, breathers or finite-gap solutions) by means of linear Fourier harmonics because they are not proper modes in complex situation. Moreover, such linear methods are not adequate in more complicated situations which are very far even from the exactly integrable case (Liouvillean tori). It would appear that as a first step in this direction is to find a reasonable extension of understanding of the non-equilibrium dynamics as a whole. One needs to sketch up the underlying ingredients of the theory (spaces of states, observables, measures, classes of smoothness, dynamical set-up, etc.) in an attempt to provide the maximally extendable but at the same time really calculable and realizable description of the complex dynamics inside hierarchies like BBKGY and their reductions. The general idea is rather simple: it is well known that the idea of "symmetry" is the key ingredient of any reasonable physical theory from classical (in)finite dimensional (integrable) Hamiltonian dynamics to different sub-planckian models based on strings. A starting point for us is a possible model for fundamental localized modes with the subsequent description of the whole zoo of possible realizable (controllable) states/patterns which may be useful from the point of view of experimentalists and engineers. The proper representation theory is well known as "local nonlinear harmonic analysis", in particular case of simple underlying symmetry-affine group-aka wavelet analysis. From our point of view the advantages of such approach are as follows:

i) natural realization of localized states in any proper functional realization of (functional) space of states,

ii) hidden symmetry of chosen realization of proper functional model provides the (whole) spectrum of possible states via the so-called multiresolution decomposition.

So, indeed, the hidden symmetry (non-abelian affine group in the simplest case) of the space of states via proper representation theory generates the physical spectrum and this procedure depends on the choice of the functional realization of the space of states. It explicitly demonstrates that the structure and properties of the functional realization of the space of states are the natural properties of physical world at the same level of importance as a particular choice of Hamiltonian, or the equation of motion, or the action principle (variational method). It should be noted that in such picture we can naturally include the effects of self-interaction on the way of construction and subsequent analysis of nonlinear models. So, our consideration will be in the framework of (Nonlinear) Pseudodifferential Dynamics (ΨDOD). Existence of such internal multiscales with different dynamics at each scale and transitions, interactions, and intermittency between scales demonstrates that statistical mechanics in BBGKY form, despite its linear structure, is really a complicated problem from the mathematical point of view. It seems, that well-known underlying "stochastic" complexity is a result of transition by means of (still rather unclear) procedure of dynamical irreversible evolution or interscale redistribution from complexity related to individual classical dynamics to the rich pseudodifferential (more exactly, microlocal) structure on the non-equilibrium ensemble side. Anyway, the whole zoo of solutions consists of possible patterns, including very important ones from the point of view of underlying physics:

iii) chaotic states (definitely, non proper for modeling of fusion state but proper for pre- or post-fusion description) vs. localized modes (basis modes, eigenmodes) and fusion-like localized patterns constructed from them by means of proper representation of underlying hidden symmetry group. For practical reasons controllable patterns (with prescribed behaviour) are the most useful. We mention the so-called waveleton-like pattern which we consider as the most important one. It means:

iv) waveleton $\approx$ (meta)stable localized (controllable) pattern.

To summarize, the approach described below allows

v) to solve wide classes of general ΨDOD problems, including generic for us BBGKY hierarchy and its reductions, and

vi) to present the analytical/numerical realization for physically interesting patterns, like fusion states.

3 Set-up

Let us consider the following generic ΨDOD dynamical problem

$$L^j\{Op^i\}\Psi = 0, \tag{1}$$

described by a finite or infinite number of equations which include general classes of operators Op^i such as differential, integral, pseudodifferential etc. Surely, all hierarchies and their reductions are inside this class. The main objects are:

i) (Hilbert) space of states, $H = \{\Psi\}$, with a proper functional realization, e.g.,: L^2, Sobolev, Schwartz, C^0, C^k, ... C^∞, ...; definitely, $L^2(R^2)$, $L^2(S^2)$, $L^2(S^1 \times S^1)$, $L^2(S^1 \times S^1 \ltimes Z_n)$ are different objects proper for different physics inside. E.g., two last cases describe tokamak and stellarator, correspondingly. Of course, they are different spaces and generate different physics.

ii) Class of smoothness. The proper choice determines natural consideration of dynamics with/without Chaos/Fractality properties.

iii) Decompositions

$$\Psi \approx \sum_i a_i e^i \tag{2}$$

via high-localized bases (wavelet families, generic wavelet packets etc), frames, atomic decomposition (Fig. 1, 2) with the following main properties: (exp) control of convergence, maximal rate of convergence for any Ψ in any H [5], [6].

iv) Observables/Operators (ODO, PDO, ΨDO , SIO,..., Microlocal analysis of Kashiwara-Shapira (with change from functions to sheaves)) satisfy the main property - the matrix representation in localized bases

$$< \Psi|Op^i|\Psi > \tag{3}$$

is maximum sparse.

$$\begin{pmatrix} D_1 & 0 & 0 & \dots \\ 0 & D_2 & 0 & \dots \\ 0 & 0 & D_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \text{where} \qquad D_i = \begin{pmatrix} A_i & B_i \\ C_i & 0 \end{pmatrix}.$$

Such almost diagonal structure is provided by the so-called Fast Wavelet Transform [5].

v) Measures: multifractal wavelet measures $\{\mu_i\}$ together with the class of smoothness are very important for analysis of complicated analytical behaviour [5].

vi) Variational/Projection methods, from Galerkin to Rabinowitz minimax, Floer (in symplectic case of Arnold-Weinstein curves with preservation of Poisson or symplectic structures). Main advantages are the reduction to algebraic systems, which provides a tool for the smart subsequent control of behaviour and control of convergence.

vii) Multiresolution or multiscale decomposition, MRA (or wavelet microscope) consists of the understanding and choosing of (internal) symmetry structure, e.g., affine group = {translations, dilations} or many others; construction of representation/action of this symmetry on $H = \{\Psi\}$. As a result of such hidden coherence together with using point vi) we'll have: **a).** Localized Bases, **b).** Exact Multiscale Decomposition with the best convergence properties and real evaluation of the rate of convergence via proper "multi-norms".

Fig. 4 demonstrates MRA decomposition for the kick (Fig. 3).

viii) Effectiveness of proper numerics: CPU-time, HDD-space, minimal complexity of algorithms, and (Shannon) entropy of calculations are provided by points i)-vii) above.

Finally, such Variational-Multiscale approach based on points i)-viii) provides us with the full possible Zoo of Patterns: localized, chaotic, etc. In next Sections we will consider details for important case of kinetic equations.

4 Description

So, we will consider the application of our numerical/analytical technique based on local nonlinear harmonic analysis approach for the description of complex non-equilibrium behaviour of statistical ensembles, considered in the framework of the general BBGKY hierarchy of kinetic equations, including quantum counterpart, and their different truncations/reductions [7]–[23]. The main points of our ideology are described below. All these facts are well-known or mentioned above but it is preferable to bring it together to present our arguments in most clear form.

• Kinetic theory in nonequilibrium situation is an important part of general statistical physics related to phenomena which cannot be understood on the thermodynamical or fluid models level of description as well as on the level of numerical modeling based on consideration of "collection of particles" instead of "complex non-equilibrium ensembles of particles".

• We restrict ourselves to the rational/polynomial type of nonlinearities (with respect to the set of all dynamical variables, including partitions) that allows to use our results, based on the so called multiresolution framework and the variational formulation of initial nonlinear (pseudodifferential) problems.

• Our approach is based on the set of mathematical methods which give a possibility to work with well-localized bases in functional spaces and provide the maximum sparse forms for the general type of operators (differential, integral, pseudodifferential) in such bases.

• It provides the best possible rates of convergence and minimal complexity of algorithms inside and, as a result, saves CPU time and HDD space.

• In all cases below by the system under consideration we mean the full BBGKY hierarchy or some its cut-off or its various reductions. Our scheme of cut-off for the infinite system of equations is based on some criterion of convergence of the full solution by means of some norm introduced in the proper functional space constructed by us.

• This criterion is based on a natural norm in the proper functional space, which takes into account (non-perturbatively) the underlying multiscale structure of complex statistical dynamics. According to our approach the choice of the underlying functional space is important to understand the corresponding complex dynamics.

• It is obvious that we need accurately to fix the space, where we construct the solutions, evaluate convergence, etc. and where the very complicated infinite set of operators, appeared in the BBGKY formulation, acts.

• We underline that many concrete features of the complex dynamics are related not only to the concrete form/class of operators/equations but depend also on the proper choice of function spaces, where operators act. It should be noted that the class of smoothness (related at least to the appearance of chaotic/fractal-like type of behaviour) of the proper functional space under consideration plays a key role in the following.

• At this stage our main goal is an attempt of classification and construction of a possible zoo of nontrivial (meta) stable states/patterns: high-localized (nonlinear) eigenmodes, complex (chaotic-like or entangled) patterns, localized (stable) patterns (waveletons). We will use it later for fusion description, modeling and control.

• Localized (meta)stable pattern (waveleton) is a good image for fusion state in plasma (energy confinement).

Our constructions can be applied to the following general individual (members of ensemble under consideration) Hamiltonians:

$$H_N = \sum_{i=1}^{N}\Big(\frac{p_i^2}{2m} + U_i(q)\Big) + \sum_{1\le i\le j\le N} U_{ij}(q_i,q_j), \tag{4}$$

where the potentials $U_i(q) = U_i(q_1,\dots,q_N)$ and $U_{ij}(q_i,q_j)$ are restricted to rational functions on the coordinates. Let L_s and L_{ij} be the Liouvillean operators and

$$F_N(x_1,\dots,x_N;t) \tag{5}$$

be the hierarchy of N-particle distribution function, satisfying the standard BBGKY hierarchy (υ is the volume):

$$\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{\upsilon}\int d\mu_{s+1}\sum_{i=1}^{s} L_{i,s+1}F_{s+1}. \tag{6}$$

Our key point is the proper nonperturbative generalization of the previous perturbative multiscale approaches (like Bogolubov/virial expansions). The infinite hierarchy of distribution functions is:

$$\begin{aligned}
&F = \{F_0, F_1(x_1;t),\dots,F_N(x_1,\dots,x_N;t),\dots\},\\
&F_p(x_1,\dots,x_p;t)\in H^p,\quad H^0 = R,\ H^p = L^2(R^{6p}),\\
&F\in H^\infty = H^0\oplus H^1\oplus\dots\oplus H^p\oplus\dots
\end{aligned} \tag{7}$$

with the natural Fock space like norm (guaranteeing the positivity of the full measure):

$$(F,F) = F_0^2 + \sum_i \int F_i^2(x_1,\dots,x_i;t)\prod_{\ell=1}^{i}\mu_\ell. \tag{8}$$

• Multiresolution decomposition (filtration) naturally and efficiently introduces the infinite sequence (tower) of the underlying hidden scales, which is a sequence of increasing closed subspaces $V_j \in L^2(R)$:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \tag{9}$$

• Our variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraic problems at the second one.

Let L be an arbitrary (non)linear differential/integral operator with matrix dimension d (finite or infinite), which acts on some set of functions from $L^2(\Omega^{\otimes^n})$: $\Psi \equiv \Psi(t, x_1, x_2, \dots) = \Big(\Psi^1(t, x_1, x_2, \dots), \dots, \Psi^d(t, x_1, x_2, \dots)\Big)$, $x_i \in \Omega \subset \mathbf{R}^6$, n is the number of particles:

$$\begin{aligned} L\Psi &\equiv L(Q, t, x_i)\Psi(t, x_i) = 0, \\ Q &\equiv Q_{d_0,d_1,d_2,\dots}(t, x_1, x_2, \dots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \dots, \int \mu_k) \\ &= \sum_{i_0,i_1,i_2,\dots=1}^{d_0,d_1,d_2,\dots} q_{i_0 i_1 i_2 \dots}(t, x_1, x_2, \dots)\Big(\frac{\partial}{\partial t}\Big)^{i_0}\Big(\frac{\partial}{\partial x_1}\Big)^{i_1}\Big(\frac{\partial}{\partial x_2}\Big)^{i_2}\dots\int \mu_k. \end{aligned} \tag{10}$$

Let us consider now the N mode approximation for the solution as the following ansatz:

$$\Psi^N(t, x_1, x_2, \dots) = \sum_{i_0,i_1,i_2,\dots=1}^{N} a_{i_0 i_1 i_2 \dots} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \dots (t, x_1, x_2, \dots). \tag{11}$$

We shall determine the expansion coefficients from the following conditions:

$$\ell^N_{k_0,k_1,k_2,\dots} \equiv \int (L\Psi^N) A_{k_0}(t) B_{k_1}(x_1) C_{k_2}(x_2) \mathrm{d}t \mathrm{d}x_1 \mathrm{d}x_2 \dots = 0. \tag{12}$$

As a result the solution has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes

$$\begin{aligned} F(t, x_1, x_2, \dots) &= \sum_{(i,j)\in Z^2} a_{ij} U^i \otimes V^j(t, x_1, x_2, \dots), \\ V^j(t) &= V_N^{j,slow}(t) + \sum_{l \ge N} V^j_l(\omega_l t), \ \omega_l \sim 2^l, \\ U^i(x_s) &= U_M^{i,slow}(x_s) + \sum_{m \ge M} U^i_m(k^s_m x_s), \ k^s_m \sim 2^m, \end{aligned} \tag{13}$$

These formulas give the expansion into a slow and fast oscillating parts. So, we may move from the coarse scales of resolution (coarse graining) to the finest ones for obtaining more detailed information about the dynamical process.

• In this way one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode.

• It should be noted that such representations give the best possible localization properties in the corresponding (phase) space/time coordinates.

• Numerical calculations are based on compactly supported wavelets and related wavelet families and on evaluation of the accuracy on the level N of the corresponding cut-off of the full system w.r.t. the norm (8):

$$\|F^{N+1} - F^N\| \le \varepsilon \tag{14}$$

• Numerical modeling shows the creation of various internal structures from localized modes, which are related to (meta)stable or unstable type of behaviour and the corresponding patterns (waveletons) formation. Reduced algebraic structure provides the pure algebraic control of stability/unstability scenario.
• So, we considered the construction for controllable (meta) stable waveleton configuration representing a reasonable approximation for the possible realizable confinement state.

5 Conclusions

Let us summarize our main results:

Physical Conjectures:

P1 State of fusion (confinement of energy) in plasma physics may and need be considered from the point of view of non-equilibrium statistical physics. According to this BBGKY framework looks naturally as first iteration. Main dynamical variables are partitions.

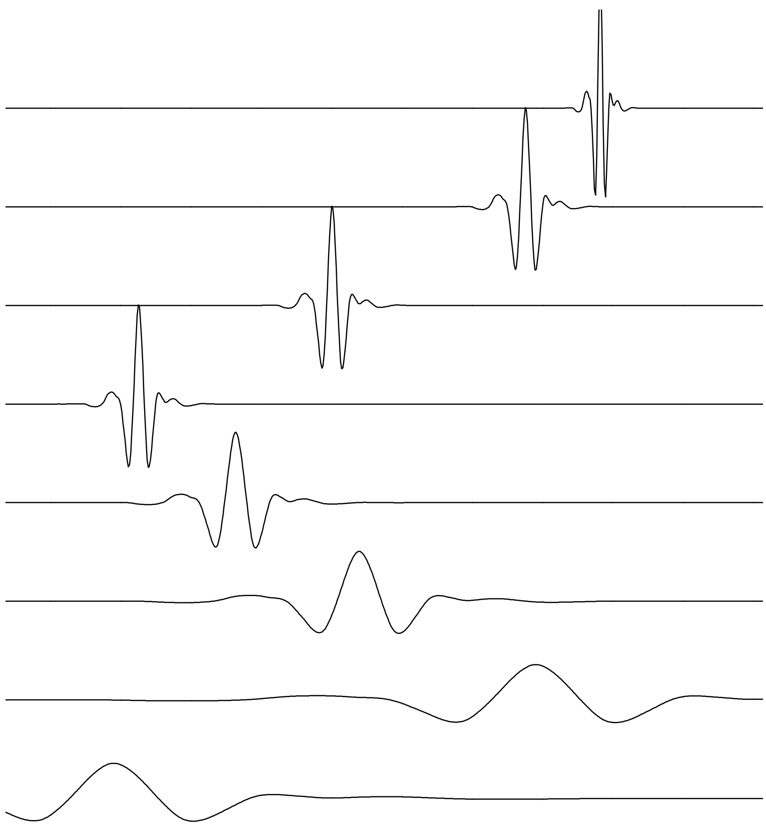

Figure 1: Localized Modes.

P2 High localized nonlinear eigenmodes (Figs. 1, 2) are real physical modes important for fusion modeling. Intermode multiscale interactions create various patterns from these fundamental building blocks, and determine the behaviour of plasma (Fig. 5). High localized (meta) stable patterns (waveletons), considered as long-living fluctuations, are proper images for plasma in fusion state (Fig. 6).

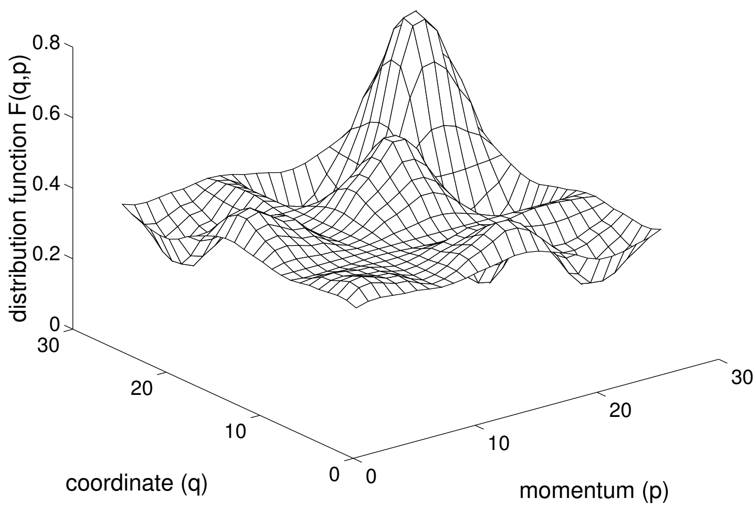

Figure 2: Two-dimensional Localized Mode Contribution to Distribution Function.

Mathematical framework:

M1 The problems under consideration, like BBGKY hierarchies (6) or their reductions from paper [3] are considered as ΨDO problems in the framework of proper family of methods unified by effective multiresolution approach or local nonlinear harmonic analysis on the orbits of representations of hidden underlying symmetry of properly chosen functional space.

M2 Formulae (13) based on generalized dispersion relations (GDR) (12) provide exact multiscale representation for all dynamical variables (partitions, first of all) in the basis of high-localized nonlinear (eigen)modes. Numerical realizations in this framework are maximally effective from the point of view of complexity of all algorithms inside. GDR provide the way for the state control on the algbraic level.

Realizability

According to this approach, it is possible on formal level, in principle, to control ensemble behaviour and to realize the localization of energy (confinement state) inside the waveleton configurations created from a few fundamentals modes only during self-organization via possible (external) control (Fig. 7, 8).

Open Questions

Q1 Definitely, all above is only very naive ensemble approach. Current level of non-equilibrium statistical physics provides us only by BBGKY generic framework. All related internal unsolved problems are well-known but we still have nothing better. At the same time possible Vlasov-like reductions or phenomenological models also look as very far from reasonable from the point of view of the fusion problem set-up.

Q2 Considering for allusion successful areas of physics like superconductivity, for example, we may conclude that only microscopic BCS formulation provides the full explanation although Ginzburg-Landau (GL) phenomenological approach and even Froelich's and London's ones contributed to the general picture. Whether Vlasov equations are the analogue of GL ones and whether it is possible to construct microscopic model for plasma, these two important questions remain unanswered at present time.

Q3 It may be natural also that approaches proposed in this paper and related ones are wrong because the proper and adequate framework for solution of fusion problem is re-

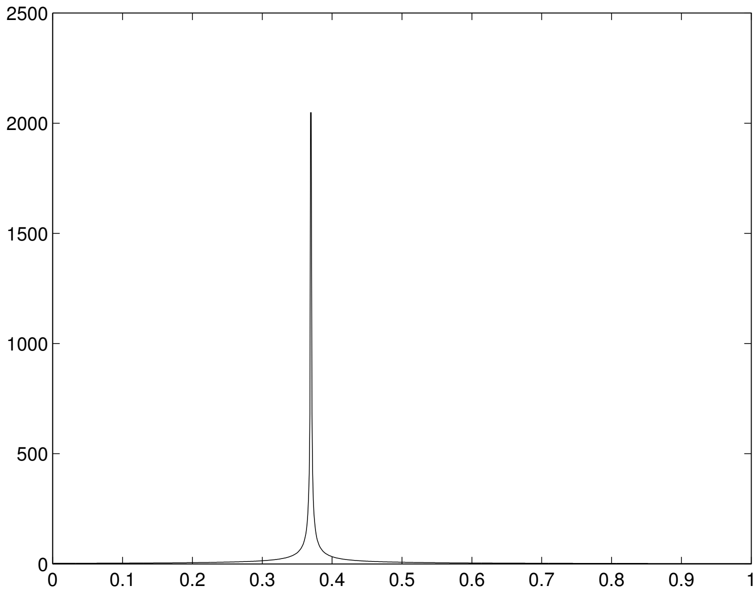

Figure 3: Kick.

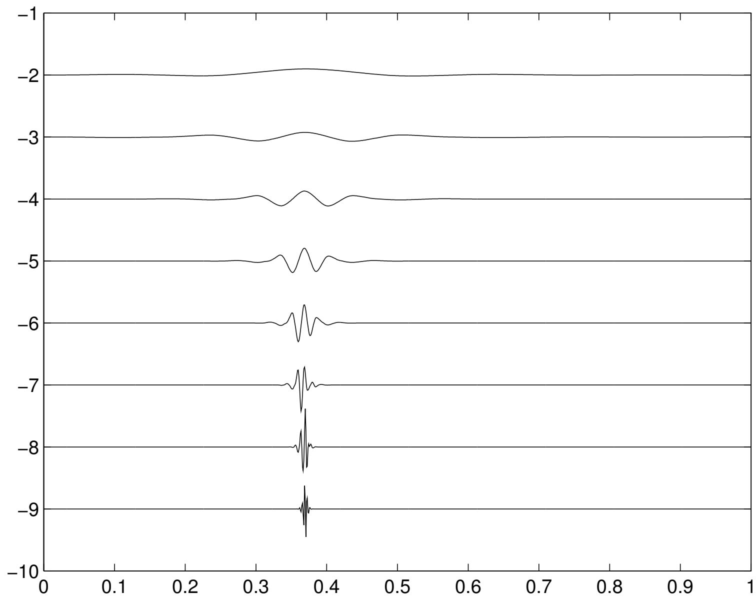

Figure 4: MRA for Kick.

lated to confinement of magnetic lines or loops (new physical dynamical variables instead of partitions) or fluxes instead of confinement of localized point modes (attribute of any local field theory) considered as new and really proper physical variables (magnetic reconnection problem can be considered in the same framework). Such an approach demands the topological background related to proper mathematical constructions. As allusion it is possible to consider, e.g., the description of (fractional) quantum Hall effect by means of Chern-Simons/anyon models which allow to describe the dynamics on (of) knots and braids analytically. Anyway, it is still possible to apply successfull methods from (M1) and (M2) here too. Other open possibility is related to taking into account internal quantum properties. From this point of view our approach is very useful because we unify quantum description and its classical counterpart in the general ΨDO framework [4]. We

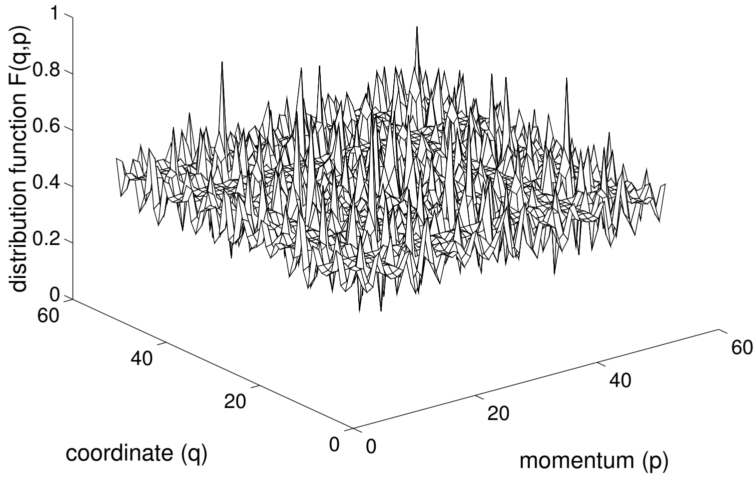

Figure 5: Trash: Chaotic Partition.

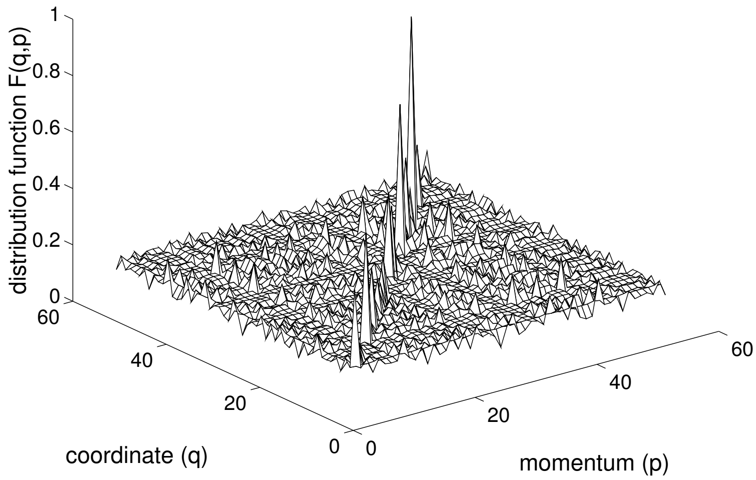

Figure 6: Goal: Localized Partition.

believe that the appearance of nontrivial localized (meta) stable patterns (Fig. 7, 8) observed by these methods is a general effect which present in the full BBGKY-hierarchy, due to its complicated intrinsic multiscale dynamics and it depends on neither the cut-off level nor the phenomenological-like hypothesis on correlators. So, representations for solutions like (13) and as a result the prediction of the existence of the (meta) stable localized patterns/states (waveletons) which can realize energy confinement (fusion) states in BBGKY-like systems are the main results of this approach. In addition, such an approach open the way to solve the control problem by means of reduction from initial (pseudodifferential) formulation to reduced set of algebraic one (12) and as a result to create and support the needed fusion state(s) after solution of proper control problem.

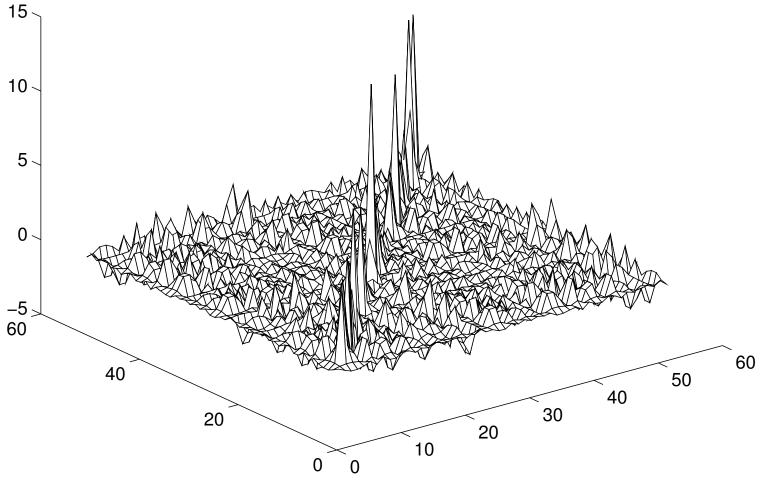

Figure 7: Fusion-like state: Confinement of Energy.

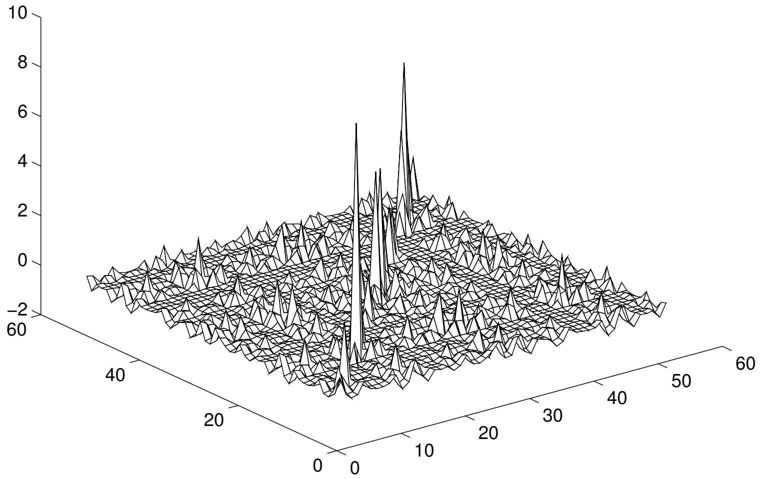

Figure 8: Fusion-like state: Waveleton Pattern.

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