Multiscale representation for space-charge dominated beam transport

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Outline

Abstract

We discuss space-charge dominated beam transport systems, where space-charge forces have the same order as external focusing forces and the dynamics of the corresponding emittance growth. We consider the coherent modes of oscillations and coherent instabilities both in the different nonlinear envelope models and in initial collective dynamics picture described by the Vlasov system. Our calculations are based on the variation approach and multiresolution analysis in the base of high-localized generalized (coherent) states. We control contributions to the dynamical evolution from the underlying tower of hidden multiscales via invariant orbital nonlinear eigenmode expansions in the base of compactly supported wavelets and wavelet packets bases.

In this paper, we consider the applications of our numerical-analytical technique [3]–[14] based on the Local Nonlinear Harmonic Analysis (LNHA) approach for calculations related to the description of different collective space-charge effects in high-intensity beam dynamics, which are very important both in accelerator and plasma physics [1], [2]. We consider models for space-charge dominated beam transport systems in case when space-charge forces are the same order as external focusing forces and the dynamics of the corresponding emittance growth is related to oscillations of underlying coherent modes [2].
Such an approach may be useful in all models where there exists a reasonable reduction from complicated problems related to the collective type of behaviour inside the corresponding statistical ensembles to the more formal problems described by systems of regular nonlinear ordinary/partial differential equations. It should be noted that other our approaches can be applied successfully to the microscopical models based on the full hierarchy of kinetic equations as well as to important and well-known reductions like various versions of the full Vlasov-Maxwell-Poisson systems, but, definitely, the phenomenological models described here are also important in the physical modeling.

We consider a framework based on the second moments of distribution functions for the calculation of the evolution of the very important in applications rms envelope of a beam [2].
The rational type of nonlinearities allows us to use our technique developed in [3]-[14] and based on the application of LNHA ideology to the variational formulation of initial nonlinear problems. LNHA is a set of mathematical methods, which provides a possibility to work with the best possible well-localized bases in any functional spaces and even the scale of spaces and allows to obtain the maximum sparse (i.e., effective) form for the general type of operators like differential, integral, pseudododifferential in such bases.

In part 2, we describe a model based on the Vlasov-type description and corresponding generic rms equations.

In part 3, we present explicit analytical constructions for solutions of the Vlasov (beyond the trivial smooth gaussian part of spectrum) and rms equations described in part 2, which are based on the variational formulation of initial dynamical problems, multiresolution representation, and very effective (much more than other ones, e.g., SVD) fast wavelet transform technique [15].

We provide the explicit representation for all dynamical variables in the base of the high-localized generalized coherent states described by natural and realistic nonlinear eigenmodes like proper wavelet packets. Our solutions are parametrized by the solutions of a number of the reduced algebraical problems, one from which is nonlinear with the same degree of nonlinearity as the initial ones and the others are the linear problems which come from the corresponding multiresolution constructions.
The former one encodes the data of physical spectrum. We name it the Generalized Dispersion Relations (GDR) because it is the natural generalization of the abelian Fourier–based counterpart of our noncommutative internal hidden symmetry Harmonic Analysis we use here systematically, while the latter ones provide information about the refined analytical aspects like a class of smoothness and fine details (more exactly, the whole tower of details) of the geometry of the underlying functional spaces and operators there.

In part 4, we present the results of numerical calculations. ifferential in such bases.

Vlasov/RMS equations

Let $\mathbf{x}(s) = \big(x_1(s), x_2(s)\big)$ be transverse coordinates, then the single-particle equation of motion is (refs.[1], [2] for designations):

$$\mathbf{x}'' + \mathbf{k}_x(s)\mathbf{x} - D\mathbf{E}_x(\mathbf{x}, s) = 0, \tag{1}$$

where $D = q/m\gamma^3 v_0^2$, $\mathbf{k}_x(s)$ describes the periodic focusing force and $\mathbf{E}$ satisfies the Poisson equation:

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0} n(\mathbf{x}, s) \tag{2}$$

with the following density projection:

$$n = \int\int f(\mathbf{x}, \mathbf{x}', s)\mathrm{d}\mathbf{x}'. \tag{3}$$

The distribution function $f(\mathbf{x}, \mathbf{x}')$ satisfies the Vlasov equation

$$\frac{\partial f}{\partial s} + (\mathbf{x}' \cdot \nabla)f - \Big(\mathbf{k} - D\mathbf{E}\Big) \cdot \nabla_{\mathbf{x}'} f = 0 \tag{4}$$

Using standard procedure, which takes into account that rms emittance is described by the seconds moments only

$$\varepsilon^2_{x,rms} =< x_i^2 >< x_j^2 > - < x_i x_j >^2 \; (i \neq j), \tag{5}$$

we arrive to the beam envelope equations for σ_{x_i}:

$$\begin{aligned} \sigma''_{x_1} + k_{x_1}(s)\sigma_{x_1} - \varepsilon^2_{x_1}/\sigma^3_{x_1} - C/(\sigma_{x_1} + \sigma_{x_2}) = 0 \\ \sigma''_{x_2} + k_{x_2}(s)\sigma_{x_2} - \varepsilon^2_{x_2}/\sigma^3_{x_2} - C/(\sigma_{x_1} + \sigma_{x_2}) = 0, \end{aligned} \tag{6}$$

where $C = qI/\pi\varepsilon_0 m\gamma^3 v_0^2$ but only in case when we can calculate explicitly $< x_i E_j >$. An additional equation describes evolution of $\varepsilon^2_{x_i}(s)$:

$$\frac{\mathrm{d}\varepsilon^2_x}{\mathrm{d}s} = 32D\big(< x_i^2 >< x_j E_i > - < x_i x_j >< x_i E_i > \big). \tag{7}$$

To compute the nonlinear E_i terms, we need to take into account higher order moments and as a result it leads to an infinite system of equations. But, in case of any sort of subsequent approximation, from the formal point of view, these rms-type envelope equations are not more than nonlinear differential equations with rational nonlinearities and variable coefficients.

One of the most promising LNHA advantages demonstrates that for a large class of operators the high-localized nonlinear eigenmodes are the best possible approximation for true eigenvectors, and as a result the corresponding matrices of operators are almost diagonal. The fast wavelet transform [15] provides the maximum sparse form of all physically motivated operators under consideration. It is true as in case of the Vlasov-type system of equations (1)-(4) as well for appropriate reductions via moment expansions like (5)–(7). where we have both differential and integral operators inside.

So, let us denote the (integral/differential) operator involved into equations (1)-(4) or (5)-(7) as T and its kernel as K. We have the following representation:

$$< Tf, g >= \int\int K(x,y)f(y)g(x)\mathrm{d}x\mathrm{d}y. \qquad (8)$$

In case when f and g are high-localized nonlinear eigenmodes generated by actions of the internal symmetry group (e.g. non-abelian affine group of translations and dilations) of the underlying functional space, e.g., wavelets $\varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$, the expression (8) provides the standard representation of operator T. Let us consider the multiresolution representation of the underlying functional space by action of this hidden (e.g., affine) symmetry

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots \qquad (9)$$

The basis in each V_j is $\varphi_{j,k}(x)$, where indices k, j represent translations and scaling respectively or the action of the underlying affine group, which acts as a "microscope" and allows us to construct the corresponding solution with the needed level of resolution.

Let T acts as follows : $L^2(R^n) \to L^2(R^n)$, with the kernel $K(x, y)$, and let $P_j : L^2(R^n) \to V_j$ $(j \in Z)$ be projection operators on the subspace V_j corresponding to j level of resolution:

$$(P_j f)(x) = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x). \qquad (10)$$

Let $Q_j = P_{j-1} - P_j$ be a projection operator on the subspace W_j, then we have the following "microscopic or telescopic" representation of the operator T, which takes into account contributions from each level of resolution from different scales starting with coarse graining scales and ending by finest ones [15]:

$$T = \sum_{j \in Z} (Q_j T Q_j + Q_j T P_j + P_j T Q_j). \qquad (11)$$

Let us remember that this is a result of the presence of the hidden affine group inside this construction and it is exactly the generic point of the whole LNHA approach.

The non-standard form of the operator representation [15] is the representation of the operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j\in Z}$, acting on the subspaces V_j and W_j:

$$A_j: \ W_j \to W_j, \ B_j : V_j \to W_j, \ \Gamma_j : W_j \to V_j, \tag{12}$$

where operators $\{A_j, B_j, \Gamma_j\}_{j\in Z}$ are defined as $A_j = Q_j T Q_j, \quad B_j = Q_j T P_j, \quad \Gamma_j = P_j T Q_j$. The operator T admits a recursive definition via

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix}, \tag{13}$$

where $T_j = P_j T P_j$ and T_j acts on $V_j : V_j \to V_j$. It should be noted that the operator A_j describes interaction on the scale j independently from other scales, operators B_j, Γ_j describe interaction between the scale j and all coarser scales, the operator T_j is an "averaged" version of T_{j-1}. We may compute such non-standard representations of the operator for the various types of operators (including pseudodifferential). As in case of the differential operator $\mathrm{d}/\mathrm{d}x$ as in other cases, in wavelet bases we need to solve only the system of linear algebraical equations.

Let

$$r_\ell = \int \varphi(x-\ell)\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\mathrm{d}x, \ell \in Z. \tag{14}$$

Then, the representation of d/dx is completely determined by the coefficients r_ℓ or by the representation of d/dx only on the subspace V_0. The coefficients $r_\ell, \ell \in Z$ satisfy the usual system of linear algebraical equations. For the representation of the operator d^n/dx^n or integral operators we have the similar reduced linear system of equations. Then finally, we have for the (affine) action of the operator $T_j(T_j : V_j \to V_j)$ on the sufficiently smooth function f:

$$(T_j f)(x) = \sum_{k\in Z}\left(2^{-j}\sum_{\ell} r_\ell f_{j,k-\ell}\right)\varphi_{j,k}(x), \tag{15}$$

where $\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x - k)$ is wavelet basis and

$$f_{j,k-1} = 2^{-j/2}\int f(x)\varphi(2^{-j}x - k + \ell)\mathrm{d}x \tag{16}$$

are wavelet coefficients.
So, we obtain in such a way the simple natural and maximally efficient linear parametrization of the matrix representation of our operators in wavelet bases and the action of these operators on an arbitrary vector in our functional space.

After that, we may apply our approach from [3]-[14]. For constructing the solutions of the rms type equations (5)–(7) obtained from Vlasov equations (1)–(4), we also may apply our variational approach, which reduces the initial problem to the problem of the solution of functional equations at the first stage and some algebraical problems at the second stage.
Finally, we can represent the solution as the exact multiscale decomposition via the compactly supported wavelet basis.
Multiresolution representation is the second main part of our construction. After that, the solution is parameterized the solutions of two reduced algebraical problems, one is nonlinear and the others are some set of linear problems, which are obtained from one of the proper multiresolution algorithms.

So, the solutions of the constrained equations (5)-(7) have the following exact multiscale representation:

$$z(s) = z_N^{slow}(s) + \sum_{j \geq N} z_j(\omega_j s), \quad \omega_j \sim 2^j, \tag{17}$$

which corresponds to the full multiresolution expansion in all underlying scales generated by action of generic internal symmetry.
Formula (17) gives us expansion into a slow part z_N^{slow} and fast oscillating parts for arbitrary N. So, we may move from the coarse scales of resolution to the finest one to obtain more detailed information about our dynamical process. The first term in the RHS of decomposition (17) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give the contribution to our full solution from each scale of resolution or each time or space scale. It should be noted that such multiscale decomposition (17) provides the best possible localization properties in the whole phase space. This is especially important because our initial dynamical variables correspond to the distribution functions/moments of an ensemble of beam particles.

Numerical modeling

Now we present a sample of the numerical illustration of the previous analytical approach. Numerical calculations are based on compactly supported wavelets, wavelet packets, and related wavelet families [16].

Figures 1 and 2 present the strong resonance region and corresponding nonlinear coherent eigenmodes decomposition according to the representation (17). They clearly demonstrate the importance of the methods allowing us to take into account the whole fine spectrum of complex nonlinear localized dynamics overcompleted by coherent localized states (resonances).

Fig. 3 presents the 6-scale/eigenmodes evaluation for the solutions of the moment/distribution (partition function) equations (1)-(7).

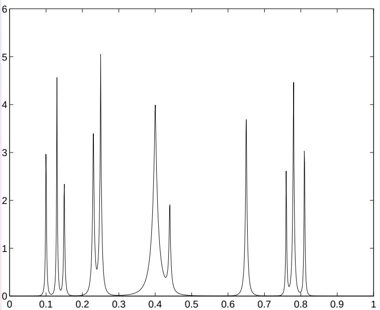

Figure 1: Region of nonlinear coherent localized states (resonances).

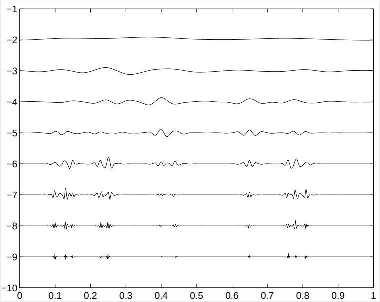

Figure 2: Coherent nonlinear eigenmodes spectrum.

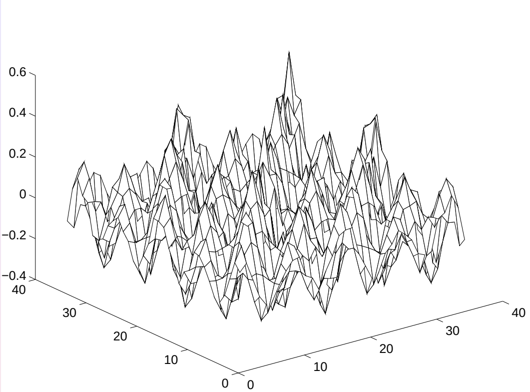

Figure 3: 6-eigenmodes decomposition.

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