# RMS envelope moment beam dynamics beyond gaussians

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APM'2014, July , 2014

# Outline

Introduction

RMS/Rate equations

Multiscale representations

Acknowledgements

## Abstract

We consider some moment type reductions from the nonlinear Vlasov- Maxwell equation to RMS/rate envelope equations for second moments related quantities. Our analysis is based on the variational- multiresolution approach for rational (in dynamical variables) approximation. It allows to control contribution to complex dynamics from each level of the resolution of the underlying hidden hierarchy of scales and represent solutions by the exact multiscale decomposition via nonlinear eigenmode expansions describing non-gaussian effects, very important in the area of high-power beam dynamics. Our approach demonstrates the advantages of the framework based on the constructing of proper well-localized bases in functional realizations of phase spaces, providing the best convergence properties of the corresponding expansions without any perturbations or/and linearization procedures and taking into account the non-gaussian features of the underlying complex dynamics.

# Introduction





In this paper, we consider the applications of a new numerical-analytical technique based on the methods of local nonlinear harmonic analysis (a.k.a. wavelet analysis in the most simple case of underlying affine symmetry) to well-known in accelerator and plasma physics nonlinear RMS (Root-Mean-Square) or Rate equations for averaged quantities related to some particular cases of nonlinear Vlasov-Maxwell equations [1]-[3].

Our starting point is a model and approach proposed by R. C. Davidson (Princeton Plasma Physics Lab), e.a. [1]. According to [1], we consider electrostatic approximation for a thin beam. This approximation is a particular important case of the general reduction from statistical collective description based on Vlasov-Maxwell equations to a finite number of ordinary differential equations for the second moments related quantities (beam radius and emittance). In our case these reduced RMS/rate equations also contain some disribution averaged quantities besides the second moments, e.g. self-field energy of the beam particles.

Such model is very efficient for analysis of many problems related to periodic focusing accelerators, e.g. heavy ion fusion and tritium production.

So, we are interested in the understanding of collective properties, nonlinear dynamics and transport processes of intense non-neutral beams propagating through a periodic focusing field.
Our approach is based on the variational-wavelet approach developed by us [4]-[15] that allows to consider rational type of nonlinearities in RMS/Rate dynamical equations containing statistically averaged quantities also.
The solution has the multiscale/multiresolution decomposition via nonlinear high-localized non-gaussian eigenmodes (waveletons) corresponding to the full multiresolution expansion in all underlying internal hidden scales.
So, we may move from coarse grained scales of resolution to the finest one to obtain more detailed information about our dynamical process.
In this way we give contribution to our full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode.
Starting from some electrostatic approximation of Vlasov-Maxwell system and RMS/rate dynamical models in Section 2, we consider the approach based on variational-wavelet formulation in Section 3.
We give explicit representation for all dynamical variables in the bases of compactly supported wavelets or nonlinear non-gaussian eigenmodes.
Our solutions are parametrized by the solutions of a number of reduced standard algebraical problems. We also present numerical modeling based on our analytical approach.

# RMS/Rate equations





In Larmor frame in thin-beam approximation with negligibly small spread in axial momentum for beam particles, we have the following electrostatic approximation for Vlasov-Maxwell equations:

$$\frac{\partial F}{\partial s} + x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} - \Big(k(s)x + \frac{\partial \psi}{\partial x}\Big)\frac{\partial F}{\partial x'} - \Big(k(s)y + \frac{\partial \psi}{\partial y}\Big)\frac{\partial F}{\partial y'} = 0$$

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\psi = -\frac{2\pi K}{N}\int \mathrm{d}x'\mathrm{d}y' F \qquad (1)$$

where $\psi(x,y,s)$ is normalized electrostatic potential and $F(x,y,x',y',s)$ is distribution function in transverse phase space $(x,y,x',y',s)$ with normalization

$$N = \int \mathrm{d}x\mathrm{d}y n, \qquad n(x,y,s) = \int \mathrm{d}x'\mathrm{d}y' F \qquad (2)$$

where $K$ is self-field perveance which measures self-field intensity [1].

Introducing self-field energy

$$E(s) = \frac{1}{4\pi K}\int \mathrm{d}x\mathrm{d}y|\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2| \qquad (3)$$

we have obvious equations for root-mean-square beam radius $R(s)$

$$R(s) = < x^2 + y^2 >^{1/2} \qquad (4)$$

and unnormalized beam emittance

$$\varepsilon^2(s) = 4(< x'^2 + y'^2 >< x^2 + y^2 > - < xx' - yy' >), \qquad (5)$$

which appear after averaging second-moments quantities regarding distribution function $F$:

$$\frac{\mathrm{d}^2 R(s)}{\mathrm{d}s^2} + \Big(k(s)R(s) - \frac{K(1+\Delta)}{2R^2(s)}\Big)R(s) = \frac{\varepsilon^2(s)}{4R^3(s)} \qquad (6)$$

$$\frac{\mathrm{d}\varepsilon^2(s)}{\mathrm{d}s} + 8R^2(s)\Big(\frac{\mathrm{d}R}{\mathrm{d}s}\frac{K(1+\Delta)}{2R(s)} - \frac{\mathrm{d}E(s)}{\mathrm{d}s}\Big) = 0, \qquad (7)$$

where the term $K(1+\Delta)/2$ may be fixed in some interesting cases, but generally we have it only as average

$$K(1+\Delta)/2 = - < x\partial\psi/\partial x + y\partial\psi/\partial y > \qquad (8)$$

regarding distribution $F$.

Anyway, the Rate equations (6), (7), (8) represent reasoanable reductions for the second-moments related quantities from the full nonlinear Vlasov-Poisson system. For trivial distributions Davidson e.a. [1] found additional reductions. For KV distribution (step-function density) the second Rate equation (7) is trivial, $\varepsilon(s)$=const and we have only one nontrivial Rate equation for RMS beam radius (6). The fixed-shape density profile ansatz for axisymmetric distributions in [1] also leads to the similar situation: emittance conservation and the same envelope equation with two shifted constants only.
So, our main goal is the search of possible solutions beyond such trivial ones. Definitely, we are interested in non-gaussian localization of fundamental eigenmodes because it can provide the way to the investigation of the underlying complex dynamics.

# Multiscale representations

According to our approach [4]-[15], which allows to find exact solutions as for full Vlasov-like systems (1), (2) as for RMS-like systems (6),(7), we avoid the standard procedure of choosing particular fixed case of distribution function $F(x, y, x', y', s)$. So, in our approach distribution/partition function(s) are independent dynamical variables which play the key role in description of ensemble dynamics, both in statistical cases and averaged ones.
Our consideration is based on the following multiscale $N$-mode anzatz:

$$F^N(x, y, x', y', s) = \sum_{i_1,\dots,i_5=1}^{N} a_{i_1,\dots,i_5} \bigotimes_{k=1}^{5} A_{i_k}(x, y, x', y', s) \quad (9)$$

$$\psi^N(x, y, s) = \sum_{j_1,j_2,j_3=1}^{N} b_{j_1,j_2,j_3} \bigotimes_{k=1}^{3} B_{j_k}(x, y, s) \quad (10)$$

Representations (9), (10) provide multiresolution representation for variational solutions of system (1), (2) [4]-[15].

Each high-localized mode/harmonics $A_j(s)$ corresponds to level $j$ of resolution from the whole underlying infinite scale of spaces:

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots,$$

where the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We present the construction of such tensor algebra based multiscales bases [16] in [15].
We will consider Rate equations (6), (7) as the following operator equation.
Let $L$, $P$, $Q$ be an arbitrary nonlinear (rational in dynamical variables) first-order matrix differential operators with matrix dimension d (d=4 in our case) corresponding to the system of equations (6), (7), which act on some set of functions
$\Psi \equiv \Psi(s) = \Big(\Psi^1(s), \ldots, \Psi^d(s)\Big), \quad s \in \Omega \subset R$ from $L^2(\Omega)$:

$$Q(R,s)\Psi(s) = P(R,s)\Psi(s) \qquad (11)$$

or

$$L\Psi \equiv L(R,s)\Psi(s) = 0 \qquad (12)$$

where $R \equiv R(s, \partial/\partial s, \Psi)$.

Let us consider now the N mode approximation for solution as the following expansion in some high-localized wavelet-like basis:

$$\Psi^N(s) = \sum_{r=1}^{N} a_r^N \phi_r(s) \tag{13}$$

We shall determine the coefficients of expansion from the following variational conditions (various related variational approaches are considered in [4]-[15]):

$$L_k^N \equiv \int (L\Psi^N)\phi_k(s)\mathrm{d}s = 0 \tag{14}$$

We have exactly $dN$ algebraical equations for $dN$ unknowns $a_r$. So, variational approach reduced the initial problem (6), (7) to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage.
As a result we have the following Reduced System of Algebraic Equations (RSAE) (or, in other words, the novel Generalized Dispersion Relations) on the set of unknown coefficients $a_i^N$ of expansion (13):

$$H(Q_{ij}, a_i^N, \alpha_I) = M(P_{ij}, a_i^N, \beta_J), \tag{15}$$

where operators $H$ and $M$ are algebraization of RHS and LHS of initial problem (11).

$Q_{ij}$ ($P_{ij}$) are the coefficients of LHS (RHS) of the initial system of differential equations (6), (7) and as consequence are coefficients of RSAE. $I = (i_1, ..., i_{q+2})$, $J = (j_1, ..., j_{p+1})$ are multiindexes by which are labelled $\alpha_I$ and $\beta_I$, the other coefficients of RSAE (15):

$$\beta_J = \{\beta_{j_1 \dots j_{p+1}}\} = \int \prod_{1 \le j_k \le p+1} \phi_{j_k}, \tag{16}$$

where $p$ is the degree of polynomial operator $P$ (11),

$$\alpha_I = \{\alpha_{i_1} \dots \alpha_{i_{q+2}}\} = \sum_{i_1,\dots,i_{q+2}} \int \phi_{i_1} \dots \dot{\phi}_{i_s} \dots \phi_{i_{q+2}}, \tag{17}$$

where $q$ is the degree of polynomial operator $Q$ (11), $i_\ell = (1, ..., q+2)$, $\dot{\phi}_{i_s} = \mathrm{d}\phi_{i_s}/\mathrm{d}s$.

According to [4]-[15] we may extend our approach to the case when we have additional constraints like (2) on the set of our dynamical variables $\Psi = \{R, \varepsilon\}$ and additional averaged terms (3), (8) also.

In this case by using the method of Lagrangian multipliers we also may apply the same approach but for the extended set of variables.

As a result, we obtain the expanded system of generic algebraical equations RSAE analogous to the system (15). Then, in the same way, we can extract from its solution the coefficients of expansion (13).

It should be noted that if we consider only truncated expansion (13) with N terms then we have the system of $N \times d$ algebraical equations with the degree $\ell = max\{p, q\}$ and the degree of this algebraical system coincides with the degree of the initial system.
So, after all we have the solution of the initial nonlinear (rational) problem (6), (7) in the form

$$R^N(s) = R(0) + \sum_{k=1}^{N} a_k^N \phi_k(s) \quad (18)$$

$$\varepsilon^N(s) = \varepsilon(0) + \sum_{k=1}^{N} b_k^N \phi_k(s), \quad (19)$$

where coefficients $a_k^N$, $b_k^N$ are the roots of the corresponding reduced algebraical (polynomial) problem RSAE (15).
Consequently, we have a parametrization of the solution of the initial problem by solution of reduced algebraical problem (15).
The problem of computations of coefficients $\alpha_I$ (17), $\beta_J$ (16) of reduced algebraical system may be explicitly solved in a number of various multiresolution approaches.

The obtained solutions are given in the form (18, (19), where $\phi_k(s)$ are proper wavelet bases functions (e.g., periodic or boundary). It should be noted that such representations provide the best possible localization properties in the corresponding (phase)space/time coordinates proper for the choice of underlying functional space.
In contrast with the different approaches formulas (18), (19) do not use perturbation technique or linearization procedures and represent dynamics via generalized nonlinear non-gaussian localized eigenmodes expansion.

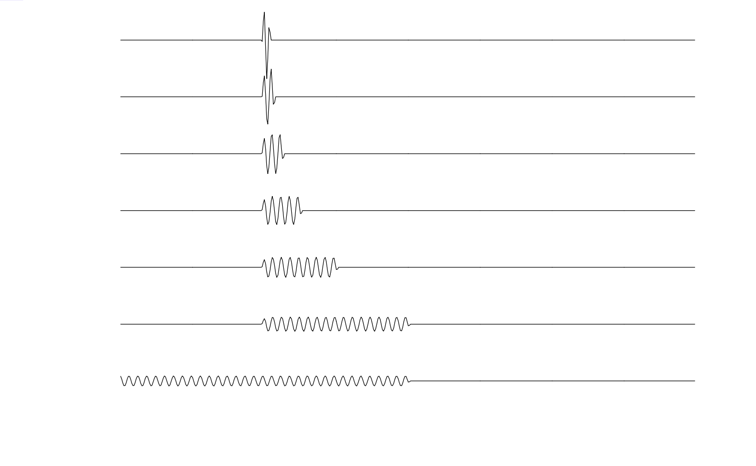

Figure: 1. Non-Linear Internal Eigenmodes.

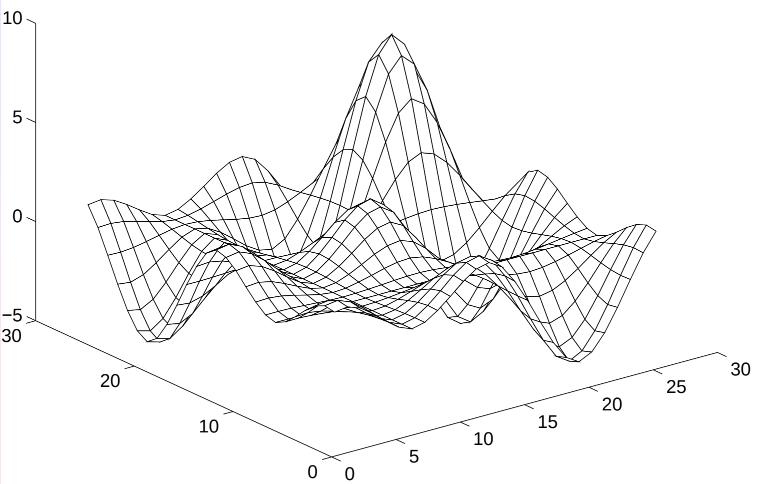


Figure: 2. Non-Gaussian Mode.

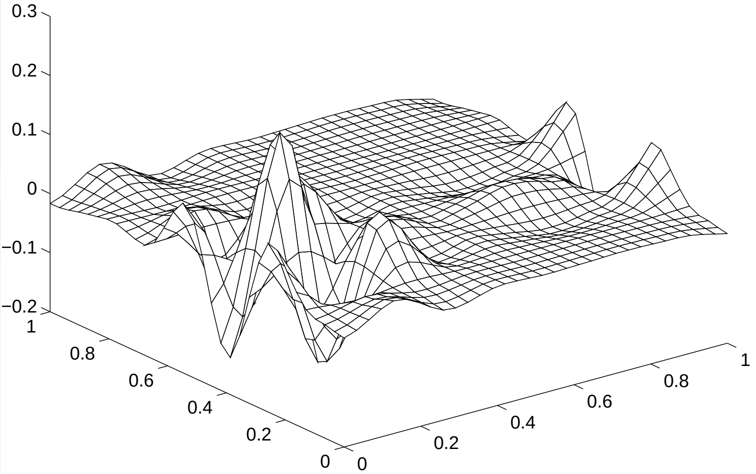


Figure: 3. Formation of Non-Gaussian Patterns.

So, our N mode construction (18), (19) gives the following general multiscale representation:

$$\begin{aligned} R(s) &= R_N^{slow}(s) + \sum_{i \geq N} R^i(\omega_i s), \quad \omega_i \sim 2^i \\ \varepsilon(s) &= \varepsilon_N^{slow}(s) + \sum_{j \geq N} \varepsilon^j(\omega_j s), \quad \omega_j \sim 2^j, \end{aligned} \tag{20}$$

where $R^i(s), \varepsilon^j(s)$ represented by a proper family of (nonlinear) eigenmodes, provide the full multiresolution/multiscale representation in the high-localized wavelet bases [17]. Such one-dimensional non-gaussian generalized internal harmonics and two-dimensional one are shown on Fig. 1 and Fig. 2 correspondingly.
The result of the full multiscale decomposition for a beam distribution function is demonstrated on Fig. 3. It is two-dimensional localized contribution to the full non-gaussian pattern.
In such a way we can construct various (meta)/(stable) patterns from high-localized (coherent) structures in spatially-extended stochastic systems with complex collective behaviour. It should be noted the importance of Generalized Dispersion Relations (14) providing the possibility for algebraical control on the zoo of possible patterns/partitions. Details will be considered elsewhere.

# Acknowledgements



We are very grateful to National Science Foundation (NSF), Institute for Pure and Applied Mathematics (IPAM) at University of California, Los Angeles, where this work was partly done, for financial support and invitation to Program "Computational Methods in High Energy Density Plasmas". We would like to thank Organizers, Professors R. Cafisch (IPAM), C. Ratsch (IPAM), F. Graziani (LLNL), M. Murillo (LANL), D. Levermore (UMD)for permanent support and patience, Prof. T. Barth (NASA) for fruitful work in his Working Group and all participants for useful discussions in different areas of Plasma Physics. We are indebted to Dr. Alexey Sergeyev (IPME/AOHGI) for his permanent help.