TWO EXPLICIT UNCONNECTED SETS OF ELLIPTIC SOLUTIONS PARAMETRIZED BY AN ARBITRARY FUNCTION FOR THE TWO-DIMENSIONAL TODA CHAIN $A_1^{(1)}$

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Two sets of explicit elliptic solutions, parametrized by an arbitrary holomorphic function are presented for the two-dimensional Toda chain $A_1^{(1)}$, obtained by means of reduction from O(3) and O(2,1) σ-models, having elliptic solutions, parametrized by an arbitrary function. A particular case – the sinh-Gordon equation is considered.

1. Algebraical, geometrical and analytical aspects of the theory of non-linear differential equations have been actively investigated recently. One of the problems appearing here is the problem of obtaining solutions of corresponding equations in explicit form. As known the general solution must be parametrized by the $2n$ arbitrary functions in the case of n equations. Examples are well known: the Liouville equation, the Toda chain [1]. But in the last case, when the chain corresponds to the Cartan matrix of the Kac–Moody algebra, the solution is expressed in the form of infinite series of very complicated structure and its use is difficult for applications (for instance, in quasiclassics). It is reasonable to reject a number of arbitrary functions in the resulting expression and due to it to obtain a formula no more complicated than the Liouville formula. In the present letter such formulae are presented for the case of the simplest two-dimensional Toda chain corresponding to the Cartan matrix of the Kac–Moody algebra $A_1^{(1)}$ [2]. This result is obtained by means of the following principle with the use of the results obtained by one of the authors earlier [3]. The local correspondence between O(3) and O(2,1) σ-models and the Toda chain under consideration is obtained which enables us to recalculate the solutions of the initial chiral model into the solutions of the reduced model [3]. In its simplest case this construction enables us to obtain the formula for the solution of the Liouville equation from the instanton sectors of O(3) and O(2,1) σ-models. In the case considered in this letter the solutions of the initial chiral models, lying out of their instanton sector are recalculated. The second element of the construction is the rather large class of solutions of the initial chiral model. For the case of O(3) and O(2,1) σ-models such solutions were obtained by one of us [4]. They are the generalization of recently obtained elliptic solutions of the O(3) σ-model [5,6], lying in the class of singular harmonic mappings [7], and they are parametrized by an arbitrary holomorphic function, and this is the most significant fact for the construction presented.

2. The action of the O(3) (O(2,1)) σ-model is:

$$S = \int h(u, \bar{u})(u_z \bar{u}_{\bar{z}} + u_{\bar{z}} \bar{u}_z) \mathrm{d}^2 x \,, \qquad (1)$$

where $u = u(z, \bar{z})$ is the initial chiral field, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ in the euclidean metric, $z = x_1 + x_2$, $\bar{z} = x_1 - x_2$ in the Minkowski metric.

$$h = h_{O(3)} = (1 + u\bar{u})^{-2} \,, \qquad (2)$$

for the case of the O(3) σ-model and

$$h = h_{O(2,1)} = (u + \bar{u})^{-2} \,, \qquad (3)$$

for the case of the O(2,1) σ-model – they are metrics on the groups O(3) and O(2,1). The equation of motion is

$$hu_{z\bar{z}} + (\partial h/\partial u) u_z u_{\bar{z}} = 0 . \qquad (4)$$

As shown in ref. [3], when using some variant of the general relativity principle, it is possible to come from the field, defined on the manifold with non-trivial curvature – the initial chiral field, the lagrangian of which contains only the kinetic term, to the new dynamic variables given on the flat manifold, but having also the potential term in the lagrangian. That is if u are the solutions of the equations of motion (4), (2), (3), lying out of the instanton sector ($u = u(z), u_{\bar{z}} = 0$; $v = v(\bar{z}), v_z = 0$) and B^1, B^2 are the new dynamic variables built of the formulae

$$\exp B^1 = hu_z \bar{u}_{\bar{z}} , \quad \exp B^2 = hu_{\bar{z}} \bar{u}_z , \qquad (5)$$

they satisfy the equations of the two-dimensional Toda chain corresponding to the Cartan matrix of the Kac–Moody algebra [1,2]:

$$B^1_{z\bar{z}} \pm 2 \exp B^1 \mp 2 \exp B^2 = 0 ,$$

$$B^2_{z\bar{z}} \mp 2 \exp B^1 \pm 2 \exp B^2 = 0 . \qquad (6)$$

The upper sign corresponds to the O(3) σ-model, the lower one to the O(2,1) σ-model.

3. Let $f = f(z), f_{\bar{z}} = 0$ be an arbitrary holomorphic function, then the solution of the equation of motion of the O(3) σ-model (4), (2) is [4]:

$$u = \left(\frac{1 + k \,\mathrm{sn} \ln(f\bar{f})^{1/2}}{1 - k \,\mathrm{sn} \ln(f\bar{f})^{1/2}} \right)^{1/2} \left(\frac{f}{\bar{f}} \right)^{1/2} , \qquad (7)$$

and for the O(2,1) σ-model (4), (3):

$$u = \exp c_1 x [\mathrm{cn}(y/k) + \mathrm{i}\,\mathrm{sn}(y/k)] , \qquad (8)$$

where

$$x \quad \text{or} \quad y = \tfrac{1}{2} \ln [f(z) \bar{f}(\bar{z})]$$

$$\text{or} \quad (1/2\mathrm{i}) \ln [f(z)/\bar{f}(\bar{z})] \qquad (9)$$

(one of the two) sn, cn are elliptic Jacobi functions with parameter k, in formula (8) $k = (1 + c_1^2)^{-1/2}$. Further we shall only consider the interval $0 < k < 1$. It should be pointed out that analogously to (8), (9) in the O(3) σ-model not only one formula (7) may be written but two (see ref. [4]). Formulae (7)–(9) and the following are written for the euclidean case, in the Minkowski case the solution is parametrized by the two arbitrary real functions depending on the light-cone variables instead of one holomorphic function.

4. If the formulae (7)–(9) are inserted in expression (5) we shall obtain the following formulae for the solutions of the Toda chain (6):

$$\exp B^1 = \tfrac{1}{16} [k \,\mathrm{cn}(\tfrac{1}{2} \ln f\bar{f}) + \mathrm{dn}(\tfrac{1}{2} \ln f\bar{f})]^2 f_z \bar{f}_{\bar{z}} / f\bar{f} , \qquad (10)$$

$$\exp B^2 = \tfrac{1}{16} [k \,\mathrm{cn}(\tfrac{1}{2} \ln f\bar{f}) - \mathrm{dn}(\tfrac{1}{2} \ln f\bar{f})]^2 f_z \bar{f}_{\bar{z}} / f\bar{f} ,$$

generated by the O(3) σ-model and

$$\exp B^1 = \frac{[c_1 + k^{-1} \mathrm{dn}(y/k)]^2}{16 \mathrm{cn}^2(y/k)} \frac{f_z \bar{f}_{\bar{z}}}{f\bar{f}} ,$$

$$\exp B^2 = \frac{[c_1 - k^{-1} \mathrm{dn}(y/k)]^2}{16 \mathrm{cn}^2(y/k)} \frac{f_z \bar{f}_{\bar{z}}}{f\bar{f}} , \qquad (11)$$

generated by the O(2,1) σ-model. Here $f = f(z), f_{\bar{z}} = 0$, is an arbitrary holomorphic function. The solution parametrized by the antiholomorphic function $g = g(\bar{z}), g_z = 0$ can be put down by changing $f\bar{f} \to g\bar{g}$, $f_z \bar{f}_{\bar{z}} \to g_{\bar{z}} \bar{g}_z$. In formula (11) y is given by formula (9). So we obtained two unconnected sets of solutions for the Toda chain (6). There arises the question of the number of such formulae.

5. Let us consider the formulae for a particular case, that is the sinh-Gordon equation. Imposing the conditions $B_1 = -B_2 = B$ leads to constraints on the arbitrary function $f(z)$ in formulae (10), (11), respectively,

$$f_z \bar{f}_{\bar{z}} / f\bar{f} = 16/(1 - k^2) , \qquad (12)$$

and

$$f_z \bar{f}_{\bar{z}} / f\bar{f} = 16 . \qquad (13)$$

Then the solution of the sinh-Gordon equation

$$B_{z\bar{z}} + 4 \sinh B = 0$$

has the form

$$\sinh B = [2k/(1 - k^2)] \,\mathrm{cn}\, \Phi \,\mathrm{dn}\, \Phi , \qquad (14)$$

obtained from the O(3) σ-model and

$$\sinh B = (2c_1/k) \mathrm{dn}(\Phi/k) / \mathrm{cn}^2(\Phi/k) , \qquad (15)$$

obtained from the O(2,1) σ-model, where in (14) $a\bar{a} = 16/(1 - k^2)$, in (15) $a\bar{a} = 16$ and

$$\Phi = \tfrac{1}{2}(az + \bar{a}\bar{z} + b + \bar{b}) ,$$

or

$$\Phi = (1/2\mathrm{i})(az - \bar{a}\bar{z} + b - \bar{b}) ,$$

where b is a complex constant. There also appears the question of classification of all the formulae of type (14), (15). It should be pointed out that the constraints (12), (13) can be looked upon not as equations on the function $f(z)$, but as equations on the curves in the z-plane *along which* the formulae for the solutions of the sinh-Gordon equation obtained from (10), (11) satisfy the equation without constraints on the function: the formulae for the solutions are so to say parametrized by the curve in the z-plane (see also ref. [3] for the case of the Dodd–Bullough equation and ref. [4]). It should be pointed out that when obtaining the formulae (14), (15) we get from (12), (13) $f(z) = \exp(az + b)$. It leads to the conclusion that formulae (7)–(9) are not appropriate for the case of singular harmonic mappings considered in refs. [5,7]. It should also be noted that the number of arbitrary holomorphic functions by which the solution is parametrized (10), (11) can be increased by means of some natural procedure of complexification [8]. It should be pointed out that close results are also obtained for the Ernst equation [4], where a more detailed consideration of some questions can be found.

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References

[1] A.N. Leznov and M.V. Saveliev, Comm. Math. Phys. 74 (1980) 111.
[2] J. Lepowsky and R.L. Wilson, Comm. Math. Phys. 62 (1978) 43.
[3] M.G. Zeitlin, Zap. Nauchn. Semin. LOMI (Leningrad) 101 (1981) 194;
A.A. Bytsenko and M.G. Zeitlin, Phys. Lett. 88A (1982) 275; Teor. Mat. Fiz. 52 (1982) 63.
[4] M.G. Zeitlin, Teor. Mat. Fiz. 57 (1983) 238;
V.E. Chelnokov and M.G. Zeitlin, Phys. Lett. 99A (1983) 147.
[5] G. Ghika and M. Visinescu, Z. Phys. C11 (1982) 353.
[6] R.B. Abbott, Z. Phys. C15 (1982) 51.
[7] J. Eells and L. Lemaire, Bull. London Math. Soc. 10 (1978) 1.
[8] V.E. Chelnokov and M.G. Zeitlin, submitted to J. Phys. A.