Wigglers: nonlinearities in multiscales. From smart storage rings to synchrotron radiation in pulsar wind nebulae

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Abstract

We consider dynamics of high-energy beams in storage rings in the presence of external insertion devices like wigglers and undulator magnets which provide: (a) an additional damping of betatron and synchrotron oscillations to create a smart beam storage system, (b) the generation of high-power synchrotron radiation, very important as for a lot of applications as well for the understanding of a number of (astrophysical) phenomena, e.g. the evolution of pulsar wind nebulae. Our machinery is based on applications of our variational–multiscale approach for the analytical/numerical treatment of the effects of insertion devices on beam dynamics. We consider the dynamical models which have polynomial nonlinearities and variable coefficients. Our approach provides all dynamical variables by exact multiscale decomposition on the whole tower of the underlying scales starting with coarse one. The generalized dispersion relations provide, in principle, the possibility for the control of dynamics on the pure algebraical level. It is very important that the description of natural nonlinearities are considered on each scale of the whole multiscale decomposition separately in the framework of the general paraproducts technique.

1 Introduction

In this paper, we consider the applications of our numerical-analytical technique [1]-[12] which is based on the methods of Local Nonlinear Harmonic Analysis (LNHA) (or wavelet analysis in the simplest case), to the treatment of effects of insertion devices on beam dynamics. Our approach is based on a generalization of the variational-multiscale approach that allows to consider both the polynomial and rational type of nonlinearities. We present the solution via full multiresolution expansion in all time/space/phase space scales, which gives us the expansion into a slow (coarse) part and fast oscillating parts. So, we may decompose our dynamical process into coarse scales of resolution and the finest one for obtaining more detailed (full, in principle) information about our dynamical process. In this way we give contribution to our exact solution from each scale of resolution. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from all underlying high-localized modes. In Part 2 we consider initial set-up for generic dynamical problem: how we may take into the account the effects of insertion external devices, like bending magnets, wigglers, undulators, on complex beam dynamics. In Part 3 we consider general framework for construction of the explicit representation for all dynamical variables in the base of maximally high localized eigenmodes (compactly supported wavelets or wavelet packets). Then, in Part 4, we consider further extension of our previous results to the case of variable coefficients. Part 5 is devoted to very important facilities that allow to consider the natural nonlinearities on each scale of the whole multiscale decomposition separately in the framework of the general paraproducts technique. These last chapters are very important for the description of Free Electron Laser and general Synchrotron Radiation considered in details in separate publications. Finally, in Part 6 we consider some numerical experiments and perspectives based on our machinery here.

2 Effects of insertion devices on beam dynamics

Assuming a sinusoidal field variation, we may consider according to [13] the analytical treatment of the effects of insertion devices on beam dynamics. One of the major harmful aspects of the installation of insertion devices is the resulting reduction of dynamic aperture. The introduction of non-linearities leads to enhancement of the amplitude-dependent tune shifts and distortion of phase space. The nonlinear fields will produce significant effects at large betatron amplitudes such as excitation of n–order resonances. The components of the insertion device vector potential used for the derivation of equations of motion are as follows:

$$\begin{aligned} A_x &= \cosh(k_x x)\cosh(k_y y)\sin(ks)/(k\rho) \\ A_y &= k_x \sinh(k_x x)\sinh(k_y y)\sin(ks)/(k_y k\rho) \end{aligned} \tag{1}$$

with $k_x^2 + k_y^2 = k^2 = (2\pi/\lambda)^2$, where λ is the period length of the insertion device, ρ is the radius of the curvature in the field B_0. After a canonical transformation to betatron variables, the Hamiltonian is averaged over the period of the insertion device and hyperbolic functions are expanded to the fourth order in x and y (or arbitrary order). Then we have the following Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}[p_x^2 + p_y^2] + \frac{1}{4k^2\rho^2}[k_x^2 x^2 + k_y^2 y^2] \\ &+ \frac{1}{12k^2\rho^2}[k_x^4 x^4 + k_y^4 y^4 + 3k_x^2 k^2 x^2 y^2] \\ &- \frac{\sin(ks)}{2k\rho}[p_x(k_x^2 x^2 + k_y^2 y^2) - 2k_x^2 p_y xy] \end{aligned} \tag{2}$$

We have in this case also the nonlinear (polynomial with degree 3) dynamical system with variable (periodic) coefficients. After averaging motion over the magnetic period we have the following generic dynamical system:

$$\ddot{x} = -\frac{k_x^2}{2k^2\rho^2}\left[x + \frac{2}{3}k_x^2x^3\right] - \frac{k_x^2xy^2}{2\rho^2} \tag{3}$$
$$\ddot{y} = -\frac{k_y^2}{2k^2\rho^2}\left[y + \frac{2}{3}k_y^2y^3\right] - \frac{k_x^2x^2y}{2\rho^2}$$

3 Wavelet framework

The first main part of our consideration is some variational approach to this problem, which reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. Multiresolution expansion is the second main part of our construction. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of the increasing closed subspaces V_j: $...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ....$ The solution is parameterized by solutions of several reduced algebraical problems, one is nonlinear and the rest ones are some linear problems, which are obtained by the method of Connection Coefficients (CC) [14]. We use here the generic compactly supported wavelet basis. Let our multiscale wavelet expansion be:

$$f(x) = \sum_{\ell\in\mathbf{Z}} c_\ell\varphi_\ell(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} c_{jk}\psi_{jk}(x) \tag{4}$$

If $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only: $f(x) = \sum\limits_{\ell\in\mathbf{Z}} c_{J\ell}\varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let $\varphi_\ell^n = \mathrm{d}^n\varphi_\ell(x)/\mathrm{d}x^n$. We consider computation of the wavelet - Galerkin integrals. Let $f^d(x)$ be d-derivative of function $f(x)$, then we have $f^d(x) = \sum_\ell c_l\varphi_\ell^d(x)$, and values $\varphi_\ell^d(x)$ can be expanded in terms of $\varphi(x)$:

$$\varphi_\ell^d(x) = \sum_m \lambda_m\varphi_m(x) \tag{5}$$
$$\lambda_m = \int\limits_{-\infty}^{\infty} \varphi_\ell^d(x)\varphi_m(x)\mathrm{d}x$$

where λ_m are wavelet-Galerkin integrals. The coefficients λ_m are 2-term connection coefficients. In general we need to find ($d_i \geq 0$):

$$\Lambda_{\ell_1\ell_2...\ell_n}^{d_1d_2...d_n} = \int\limits_{-\infty}^{\infty} \prod \varphi_{\ell_i}^{d_i}(x)dx \tag{6}$$

For the case of degree three we need to evaluate two and three connection coefficients

$$\Lambda_\ell^{d_1 d_2} = \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_\ell^{d_2}(x)dx, \tag{7}$$

$$\Lambda^{d_1 d_2 d_3} = \int\limits_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_\ell^{d_2}(x)\varphi_m^{d_3}(x)dx$$

According to CC method [14] we use the next construction. When N in scaling equation is a finite even positive integer, the function $\varphi(x)$ has compact support contained in $[0, N-1]$. For a fixed triple (d_1, d_2, d_3) only some $\Lambda_{\ell m}^{d_1 d_2 d_3}$ are nonzero: $2-N \le \ell \le N-2, \quad 2-N \le m \le N-2, \quad |\ell - m| \le N-2$. There are $M = 3N^2 - 9N + 7$ such pairs (ℓ, m). Let $\Lambda^{d_1 d_2 d_3}$ be a M-vector, whose components are numbers $\Lambda_{\ell m}^{d_1 d_2 d_3}$. Then we have the first reduced algebraical system, where Λ satisfy the following system of equations $(d = d_1 + d_2 + d_3)$:

$$A\Lambda^{d_1 d_2 d_3} = 2^{1-d}\Lambda^{d_1 d_2 d_3} \tag{8}$$

$$A_{\ell,m;q,r} = \sum_p a_p a_{q-2\ell+p} a_{r-2m+p}$$

By moment equations we have created a system of $M + d + 1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second reduced algebraical system gives us the 2-term connection coefficients $(d = d_1 + d_2)$:

$$A\Lambda^{d_1 d_2} = 2^{1-d}\Lambda^{d_1 d_2}, \quad A_{\ell,q} = \sum_p a_p a_{q-2\ell+p} \tag{9}$$

For degree more than three we have analogous additional linear problems for generic objects (6). Solving these linear problems we obtain the coefficients of the reduced nonlinear algebraical system and after that we obtain the coefficients of wavelet expansion (4). As a result we obtain the explicit exact solution of our problem in the base of compactly supported wavelets. On Fig. 1 we present an example of the base wavelet function which satisfies some boundary conditions. In the following we consider extension of this approach to the case of arbitrary variable coefficients.

4 Variable coefficients

To cover the general treatment of possible insertion device, in addition to the model described by the nonlinear (rational) differential equations, we need to consider the extension of the previous approach to the case when we take into account any type of variable coefficients (periodic, regular or singular). We can do that rather simple: we add to our construction above an additional refinement equation, which encoded all information about variable coefficients [15]. According to our variational approach we need to compute only additional integrals of the form:

$$\int_D b_{ij}(t)(\varphi_1)^{d_1}(2^m t - k_1)(\varphi_2)^{d_2}(2^m t - k_2)\mathrm{d}x, \tag{10}$$

where $b_{ij}(t)$ are arbitrary functions of time and trial functions φ_1, φ_2 satisfy the refinement equations:

$$\varphi_i(t) = \sum_{k\in\mathbf{Z}} a_{ik}\varphi_i(2t-k) \tag{11}$$

If we consider all computations in the class of compactly supported wavelets, then only a finite number of coefficients do not vanish. To approximate the non-constant coefficients, we need to choose a different refinable function φ_3 along with some local approximation scheme:

$$(B_\ell f)(x) := \sum_{\alpha\in\mathbf{Z}} F_{\ell,k}(f)\varphi_3(2^\ell t-k), \tag{12}$$

where $F_{\ell,k}$ are suitable functionals supported in a small neighborhood of $2^{-\ell}k$ and then replace b_{ij} in (10) by $B_\ell b_{ij}(t)$. In the particular case one can take a characteristic function and thus approximate non-smooth coefficients locally. To guarantee the sufficient accuracy of the resulting approximation of (10) it is important to have the flexibility of choosing φ_3 different from φ_1, φ_2. In the case when D is some domain, we can write

$$b_{ij}(t)\mid_D = \sum_{0\le k\le 2^\ell} b_{ij}(t)\chi_D(2^\ell t-k), \tag{13}$$

where χ_D is the characteristic function of D. So, if we take $\varphi_4 = \chi_D$, which is again a refinable function, then the problem of computation of (10) is reduced to the problem of calculation of the integrals:

$$\begin{aligned} H(k_1,k_2,k_3,k_4) = H(k) = \int_{\mathbf{R}^s} \varphi_4(2^j t-k_1)\cdot \\ \varphi_3(2^\ell t-k_2)\varphi_1^{d_1}(2^r t-k_3)\varphi_2^{d_2}(2^s t-k_4)\mathrm{d}x \end{aligned} \tag{14}$$

The key point is that these integrals also satisfy some sort of refinement equation [15]:

$$2^{-|\mu|}H(k) = \sum_{\ell\in\mathbf{Z}} b_{2k-\ell}H(\ell), \qquad \mu = d_1+d_2 \tag{15}$$

This equation can be interpreted as the problem of computing an eigenvector. Thus, after all that, we reduced the problem of extension of our method to the case of variable coefficients to the same standard algebraical problem as in the preceding sections. So, the general scheme is the same one and we have only one more additional linear algebraic problem by which we can parameterize the solutions of the corresponding problem in the same way.
On Fig. 2 we present, as a sample, a toy model for insertion device and on Fig. 3 the corresponding multiscale representation via localized eigenmodes according to formula (4).

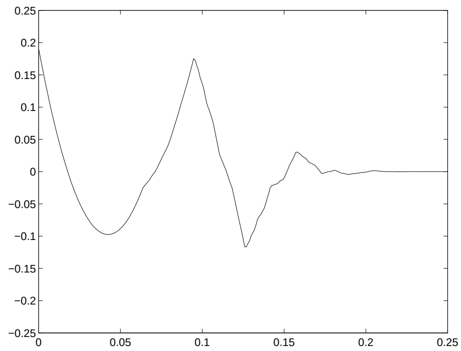

Figure 1: Base wavelet with fixed boundary conditions

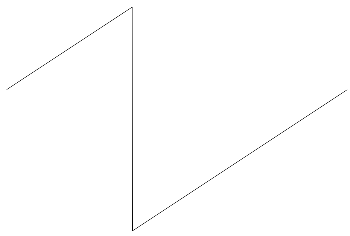

Figure 2: Sample insertion

5 Evaluation of Nonlinearities Scale by Scale

Here we present the modification of our variational–multiscale approach to the case that allows to consider separately different scales of general multiresolution. For this reason we need to compute the errors of approximations. The main problems come of course from nonlinear terms. We follow the approach from [15].
Let P_j be projection operators on the subspaces $V_j, j \in \mathbf{Z}$:

$$\begin{aligned} P_j &: L^2(\mathbf{R}) \to V_j \\ (P_j f)(x) &= \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x) \end{aligned} \tag{16}$$

and Q_j are projection operators on the subspaces W_j:

$$Q_j = P_{j-1} - P_j \tag{17}$$

So, for $u \in L^2(\mathbf{R})$ we have $u_j = P_j u$ and $u_j \in V_j$, where $\{V_j\}, j \in \mathbf{Z}$ is a multiresolution decomposition of $L^2(\mathbf{R})$. It is obviously that we can represent u_0^2 in the following form:

$$u_0^2 = 2\sum_{j=1}^{n}(P_j u)(Q_j u) + \sum_{j=1}^{n}(Q_j u)(Q_j u) + u_n^2 \tag{18}$$

In this formula there is no interaction between different scales. We may consider each term of (18) as the bilinear mappings:

$$M_{VW}^j : V_j \times W_j \to L^2(\mathbf{R}) = V_j \oplus_{j' \geq j} W_{j'} \tag{19}$$

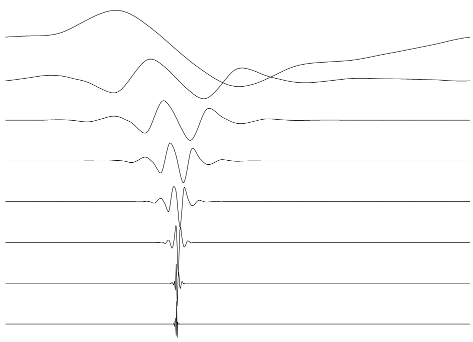

Figure 3: Multiscale representation via localized eigemodes

$$M^j_{WW} : W_j \times W_j \to L^2(\mathbf{R}) = V_j \oplus_{j' \geq j} W_{j'} \tag{20}$$

For numerical purposes we need decomposition like (18) with a finite number of scales, but when we consider limits $j \to \infty$ we will have:

$$u^2 = \sum_{j \in \mathbf{Z}} (2P_j u + Q_j u)(Q_j u), \tag{21}$$

which is para-product of Bony, Coifman and Meyer [15].
Now we need to expand (18) into the wavelet basis. To expand each term in (18) into such a wavelet basis, we need to consider the integrals of the products of the basis functions, e.g.:

$$M^{j,j'}_{WWW}(k, k', \ell) = \int_{-\infty}^{\infty} \psi^j_k(x) \psi^j_{k'}(x) \psi^{j'}_\ell(x) \mathrm{d}x, \tag{22}$$

where $j' > j$ and

$$\psi^j_k(x) = 2^{-j/2} \psi(2^{-j} x - k) \tag{23}$$

are the basis functions. If we consider compactly supported wavelets then

$$M^{j,j'}_{WWW}(k, k', \ell) \equiv 0 \quad \text{for} \quad |k - k'| > k_0, \tag{24}$$

where k_0 depends on the overlap of the supports of the basis functions and

$$|M^r_{WWW}(k - k', 2^r k - \ell)| \leq C \cdot 2^{-r\lambda M} \tag{25}$$

Let us define j_0 as the distance between scales such that for a given ε all the coefficients in (25) with labels $r = j - j'$, $r > j_0$ have absolute values less than ε. For the purposes of computing with accuracy ε we replace the mappings in (19), (20) by

$$M^j_{VW} : V_j \times W_j \to V_j \oplus_{j \leq j' \leq j_0} W_{j'} \tag{26}$$

$$M^j_{WW} : W_j \times W_j \to V_j \oplus_{J \leq j' \leq j_0} W_{j'} \tag{27}$$

Since

$$V_j \oplus_{j\leq j'\leq j_0} W_{j'} = V_{j_0-1} \tag{28}$$

and

$$V_j \subset V_{j_0-1}, \qquad W_j \subset V_{j_0-1} \tag{29}$$

we may consider bilinear mappings (26), (27) on $V_{j_0-1}\times V_{j_0-1}$. For the evaluation of (26), (27) as mappings $V_{j_0-1}\times V_{j_0-1}\to V_{j_0-1}$ we need significantly fewer coefficients than for mappings (26), (27). It is enough to consider only the coefficients

$$M(k,k',\ell) = 2^{-j/2}\int_{\infty}^{\infty}\varphi(x-k)\varphi(x-k')\varphi(x-\ell)\mathrm{d}x, \tag{30}$$

where $\varphi(x)$ is scale function. Also we have

$$M(k,k',\ell) = 2^{-j/2}M_0(k-\ell,k'-\ell), \tag{31}$$

where

$$M_0(p,q) = \int \varphi(x-p)\varphi(x-q)\varphi(x)\mathrm{d}x \tag{32}$$

Now, by the ideology of Part 3, we may derive and solve a system of linear equations to find $M_0(p,q)$ and as a result obtain the explicit representation for the solution at each scale separately.

6 Vista

We concentrated here on the general computation/analytical set-up and postponed applications to a separate paper but we need to mention that all applications, we are interested in, are covered by such a machinery. Our goals in this direction are two-fold. First of all we are interested in precise dynamics of high-energy beams in storage rings in the presence of external insertion devices like bending magnets, wigglers and undulators, which provide an additional damping of betatron and synchrotron oscillations and promise the smart beam storage systems in accelerator physics. Secondly, we hope to describe correctly the generation of high-power synchrotron radiation (including Free Electron Laser radiation), which is very important as for a lot of practical applications as well for the understanding of a number of (astrophysical) phenomena, e.g. the evolution of Crab Nebula (important experiment for understanding Lorentz violation and possible experimental signs of Quantum Gravity). In any case, the full zoo of possible patterns and coherent structures generated by hidden internal high localized modes via multiresolution representation was described by us in a companion paper in this Volume [16]. So, on the qualitative level, Figures 3 –7 [16] present all possible dynamical features, that we hope to characterize by our approach in present context: isolated high localized modes (Fig. 2 [16]), chaotic, coherent and (meta)stable localized patterns/waveletons (Figures 3–7 [16]).

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