Multiscale structure of polynomial dynamics

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Abstract

We consider a wavelet based multiscale description for nonlinear optimal dynamics (energy minimization in a high power electromechanical system as a key example). In a particular case, we have the solution as a series on shifted Legendre polynomials parametrized by the solutions of the reduced algebraical systems of equations. In the general case, we represent the solution via multiscale decomposition in the base of various families of compactly supported wavelets. In this case the solution is parametrized by solutions of two reduced algebraic problems, one as in the first case and the second one is some linear problem obtained from the popular wavelet constructions: Fast Wavelet Transform, Stationary Subdivision Schemes, the method of Connection Coefficients. Such a machinery allows us to consider maximally localized bases in the underlying functional spaces together with most sparse representation for all type of operators involving in the initial set-up. All that provides the best possible convergence properties and as a result our numerical modeling is more flexible and saves CPU time. In addition, the final representation is parametrized by the reduced pure algebraic construction (the so-called general dispersion relations) and allows us to solve the dynamical or optimal control problems (energy minimizatin, e.g.)in a most effective way.

1 Introduction

Many important physical and mechanical problems are reduced to the solving of systems of nonlinear differential equations with the polynomial type of nonlinearities. In this paper and related paper in this volume, we consider applications of methods of nonlinear local harmonic analysis (a.k.a. wavelet analysis in a simple case of affine group) to such problems. Wavelet analysis is a relatively novel set of mathematical methods, which gives us the possibility to work with well-localized bases in functional spaces and with the general type of operators (including pseudodifferential) in such bases. Many examples may be found in papers [11]–[18]. Now we apply our approach to the case of a constrained variational problem: the problem of energy minimization in electromechanical systems. We consider a synchronous electrical machine and a mill as a load (in this approach we can consider instead of the mill any mechanical load with polynomial approximation for the mechanical moment).

We consider the problem of "electrical economizer" as an optimal control problem. As result of the first stage we give the explicit time description of optimal dynamics for that electromechanical system. As a result of the second stage we give the time dynamics of our system via a construction based on the set of switched type functions (Walsh functions), which can be realized on the modern thyristor technique. In this paper, using the method of analysis of dynamical process in the Park system [1], which we developed in ref. [9], [10], we consider the optimal control problem in that system. As in [9] and [10], our goal is to construct explicit time solutions, which can be used directly in microprocessor control systems. Our consideration is based on the Integral Variational Method, which was develo ped in [21]. As we shall see later, we can obtain explicit time dependence for all dynamical variables in our optimal control problem. It is based on the fact that optimal control dynamic in our case is given by some nonlinear system of equations which is the extension of initial Park system. Moreover, the equations of optimal dynamics also is the system of Riccati type (we use the quadratic dependence of the mechanical moment). It should be noted that this system of equations is not the pure differential system but it is the mixed differential-algebraic or functional system of equations [19].

In Section 2 we consider the description of our variational approach, which can be generalized in such a way that allows us to consider it in Hamiltonian (symplectic) approach [12].

In Section 3 we consider the explicit representation for solutions. Our initial dynamical problem (without control) is described by the system of nonlinear differential equations, which has the next Cauchy form (for definitions see [9], [10])

$$\frac{di_k}{dt} = \sum_\ell A_\ell i_\ell + \sum_{r,s} A_{rs} i_r i_s + A_k(t)$$

where $A_\ell, A_{rs} (\ell, r, s = \overline{1,6})$ are constants, $A_k(t), (k = \overline{1,5})$ are explicit functions of time, $A_6(i_6, t) = a + di_6 + bi_6^2$ is analytical approximation for the mechanical moment of the mill. At initial stage of the solution of optimal control problem in both methods we need to select from initial set of dynamical variables $i_1, \dots, i_6$ the controlling and the controllable variables. In our case we consider i_1, i_2 as the controlling variables. Because we consider the energy optimization, we use the next general form of energy functional in our electromechanical system

$$Q = \int_{t_0}^{t} [K_1(i_1, i_2) + K_2(\dot{i}_1, \dot{i}_2)] dt,$$

where K_1, K_2 are quadratic forms. Thus, our functional is the quadratic functional on the variables i_1, i_2 and its derivatives. Moreover, we may consider the optimization problem with some constraints which are motivated by technical reasons [9], [10]. Then after standard manipulations from the theory of optimal control, we reduce the problem of energy minimization to some extended nonlinear system of equations. As a result, the solution of equations of optimal dynamics provides: 1). the explicit time dependence of the controlling variables $u(t) = \{i_1(t), i_2(t)\}$ which give 2). the optimum of corresponding functional of the energy and 3). explicit time dynamics of the controllable variables $\{i_3, i_4, i_5, i_6\}(t)$. The obtained solutions are given in the following form:

$$i_k(t) = i_k(0) + \sum_{i=1}^{N} \lambda_k^i X_i(t),$$

where in our first case (Section 3), we have $X_i(t) = Q_i(t)$, where $Q_i(t)$ are shifted Legendre polynomials [21] and λ_k^i are the roots of reduced algebraic system of equations. In our second case, corresponding to the generic wavelet example, considered in Section 4, the base functions $X_i(t)$ are obtained from the multiresolution decomposition in the basis of compactly supported wavelets while λ_k^i are the roots of corresponding algebraic Riccati systems with coefficients, which are given by Fast Wavelet Transform (FWT) [2] or by Stationary Subdivision Schemes(SSS) [6] or by the method of Connection Coefficients (CC) [23].
Giving the controlling variables in the explicit form, we have optimal, according to energy, dynamics in our electromechanical systems. Obviously, the technical realization of controlling variables via the arbitrary continuous functions of time is impossible, but we can replace them by their re-expansions in the basis of switching type functions, which can be realized now on the modern thyristor technique. We considered this re-expansion in [9], [10], where we used Walsh and Haar functions [3] as a base set of switching type functions. This is a special case of general sequency analysis [20]. It should be noted that the best practical realization of the expansions described in Section 4 is based on the general wavelet packet basis [4].

2 Polynomial dynamics

Our problems may be formulated as the systems of ordinary differential equations:

$$dx_i/dt = f_i(x_j, t), \ (i, j = 1, ..., n)$$

with fixed initial conditions $x_i(0)$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0 \leq t \leq 1$. Let us consider a set of functions:

$$\Phi_i(t) = x_i dy_i/dt + f_i y_i$$

and a set of the corresponding functionals:

$$F_i(x) = \int_0^1 \Phi_i(t)dt - x_i y_i \mid_0^1,$$

where $y_i(t)(y_i(0) = 0)$ are dual variables. It is obvious that the initial system and the system $F_i(x) = 0$ are equivalent. We mention here, that we can consider the symplectization of this approach (Hamiltonian version) [12] . Now we consider formal expansions for x_i, y_i:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \tag{1}$$

where, because of initial conditions, we need only $\varphi_k(0) = 0$. Then we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (1):

$$\sum_k \mu_{kr}\lambda_i^k - \gamma_i^r(\lambda_j) = 0 \tag{2}$$

Its coefficients are

$$\mu_{kr} = \int_0^1 \varphi_k'(t)\varphi_r(t)dt, \quad \gamma_i^r = \int_0^1 f_i(x_j,t)\varphi_r(t)dt.$$

Now, when we solve system (2) and determine unknown coefficients for the formal expansion (1), we therefore obtain the solution of our initial problem.
It should be noted that in case when we consider only truncated expansion (1) with N terms then we have from (2) the system of $N \times n$ algebraical equations and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial value problem for nonlinear (polynomial) system in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^{N} \lambda_i^k X_k(t), \tag{3}$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical problem (2). Consequently, we have a parametrization of solution of initial value problem by solution of reduced algebraical problem (2). But in general case, when the problem of computation of coefficients of reduced algebraical system (2) cannot be solved explicitly as in the quadratic case, which we shall consider below, we also have parametrization of solution (1) by solution of some set of the corresponding problems, which appear during calculations of the coefficients of reduced algebraic system (2).
As we shall see below, these problems may be explicitly solved in general wavelet approach.

3 Solutions: simple case

Next we consider the construction of explicit time solution for our problem. The obtained solutions are given in the form (3), where in our first case we have $X_k(t) = Q_k(t)$, where $Q_k(t)$ are shifted Legendre polynomials and λ_k^i are roots of reduced quadratic system of equations. In wavelet case $X_k(t)$ correspond to multiresolution expansions in the base of compactly supported wavelets and λ_k^i are the roots of corresponding general polynomial system (2) with coefficients, which are given by FWT, SSS or CC constructions.
According to the variational approach, we provide the reduction from the initial system of differential equations to algebraical one by means of computation of the

objects γ_a^j and μ_{ji}, which are constructed from objects:

$$\begin{aligned}
\sigma_i &\equiv \int_0^1 X_i(\tau)d\tau = (-1)^{i+1}, \qquad (4)\\
\nu_{ij} &\equiv \int_0^1 X_i(\tau)X_j(\tau)d\tau = \sigma_i\sigma_j + \frac{\delta_{ij}}{(2j+1)},\\
\mu_{ji} &\equiv \int X_i'(\tau)X_j(\tau)d\tau = \sigma_j F_1(i,0) + F_1(i,j),\\
& F_1(r,s) = [1-(-1)^{r+s}]\hat{s}(r-s-1), \quad \hat{s}(p) = \begin{cases} 1, & p \geq 0 \\ 0, & p < 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\beta_{klj} &\equiv \int_0^1 X_k(\tau)X_l(\tau)X_j(\tau)d\tau = \sigma_k\sigma_l\sigma_j + \\
& \alpha_{klj} + \frac{\sigma_k\delta_{jl}}{2j+1} + \frac{\sigma_l\delta_{kj}}{2k+1} + \frac{\sigma_j\delta_{kl}}{2l+1},\\
\alpha_{klj} &\equiv \int_0^1 X_k^* X_l^* X_j^* d\tau = \frac{1}{(j+k+l+1)R(1/2(i+j+k))} \times \\
& R(1/2(j+k-l))R(1/2(j-k+l))R(1/2(-j+k+l)),
\end{aligned}$$

if $j+k+l = 2m, m \in Z$, and $\alpha_{klj} = 0$ if $j+k+l = 2m+1$; $R(i) = (2i)!\,/(2^i i!\,)^2$, $Q_i = \sigma_i + P_i^*$, where the second equality in the formulae for $\sigma, \nu, \mu, \beta, \alpha$ hold for the first case.

4 Wavelet computations

Now we give construction for computations of objects like (4) in the generic wavelet case. We use some constructions from multiresolution analysis [8]: a sequence of successive approximation closed subspaces V_j:

$$...V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset ...$$

satisfying the following properties:

$$\bigcap_{j\in\mathbf{Z}} V_j = 0, \overline{\bigcup_{j\in\mathbf{Z}}} V_j = L^2(\mathbf{R}), \qquad f(x) \in V_j <=> f(2x) \in V_{j+1}$$

There is a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x-k)_{k\in\mathbf{Z}}\}$ forms a Riesz basis for V_0. We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbf{R})$. As usually $\varphi(x)$ is a scaling function, $\psi(x)$ is a wavelet function, where $\varphi_i(x) = \varphi(x-i)$. Scaling relation that defines φ, ψ are

$$\begin{aligned}
\varphi(x) &= \sum_{k=0}^{N-1} a_k\varphi(2x-k) = \sum_{k=0}^{N-1} a_k\varphi_k(2x),\\
\psi(x) &= \sum_{k=-1}^{N-2} (-1)^k a_{k+1}\varphi(2x+k)
\end{aligned}$$

Let be $f : \mathbf{R} \longrightarrow \mathbf{C}$ and the wavelet expansion is

$$f(x) = \sum_{\ell\in\mathbf{Z}} c_\ell \varphi_\ell(x) + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} c_{jk}\psi_{jk}(x) \tag{5}$$

The indices k, ℓ and j represent translation and scaling, respectively:

$$\varphi_{jl}(x) = 2^{j/2}\varphi(2^j x - \ell), \qquad \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k).$$

The set $\{\varphi_{j,k}\}_{k\in\mathbf{Z}}$ forms a Riesz basis for V_j. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}. Just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$. If in formulae (5) $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only

$$f(x) = \sum_{\ell\in\mathbf{Z}} c_{J\ell}\varphi_{J\ell}(x).$$

This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. We use wavelet $\psi(x)$, which has k vanishing moments

$$\int x^k\psi(x)dx = 0,$$

or equivalently

$$x^k = \sum c_\ell\varphi_\ell(x)$$

for each k, $0 \leq k \leq K$. Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0, N-1]$ and $N/2$ vanishing moments. There exists $\lambda > 0$ such that DN has λN continuous derivatives; for small $N, \lambda \geq 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi_\ell^n = d^n\varphi_\ell(x)/dx^n$. We derive the wavelet - Galerkin approximation of a differentiated $f(x)$ as:

$$f^d(x) = \sum_\ell c_l\varphi_\ell^d(x)$$

and values $\varphi_\ell^d(x)$ can be expanded in terms of $\varphi(x)$:

$$\phi_\ell^d(x) = \sum_m \lambda_m\varphi_m(x), \quad \lambda_m = \int_{-\infty}^{\infty} \varphi_\ell^d(x)\varphi_m(x)dx$$

The coefficients λ_m are 2-term connection coefficients. In general we need to find $(d_i \geq 0)$:

$$\Lambda^{d_1 d_2 \ldots d_n}_{\ell_1\ell_2\ldots\ell_n} = \int_{-\infty}^{\infty} \prod \varphi^{d_i}_{\ell_i}(x)dx \tag{6}$$

For Riccati case we need to evaluate two and three connection coefficients:

$$\Lambda_\ell^{d_1 d_2} = \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_\ell^{d_2}(x)dx, \quad \Lambda^{d_1 d_2 d_3} = \int\limits_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_\ell^{d_2}(x)\varphi_m^{d_3}(x)dx.$$

According to CC method [23] we use the next construction. When N in scaling equation is a finite even positive integer, the function $\varphi(x)$ has compact support contained in $[0, N-1]$. For a fixed triple (d_1, d_2, d_3) only some $\Lambda_{\ell m}^{d_1 d_2 d_3}$ are nonzero: $2 - N \leq \ell \leq N-2, \quad 2 - N \leq m \leq N-2, \quad |\ell - m| \leq N-2$. There are $M = 3N^2 - 9N + 7$ such pairs (ℓ, m). Let $\Lambda^{d_1 d_2 d_3}$ be an M-vector, whose components are numbers $\Lambda_{\ell m}^{d_1 d_2 d_3}$. Then we have the first key result: Λ satisfy the system of equations $(d = d_1 + d_2 + d_3)$:

$$A\Lambda^{d_1 d_2 d_3} = 2^{1-d}\Lambda^{d_1 d_2 d_3}, \quad A_{\ell,m;q,r} = \sum_p a_p a_{q-2\ell+p} a_{r-2m+p}.$$

By moment equations we have created a system of $M + d + 1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second key result gives us the 2-term connection coefficients:

$$A\Lambda^{d_1 d_2} = 2^{1-d}\Lambda^{d_1 d_2}, \quad d = d_1 + d_2, \quad A_{\ell,q} = \sum_p a_p a_{q-2\ell+p}.$$

For nonquadratic case we have analogously additional linear problems for objects (6). Also, we use FWT [2] and SSS [6] for computing coefficients of reduced algebraic systems. We use for modelling D6, D8, D10 functions and programs RADAU and DOPRI for testing [19].
As a result, we obtained the explicit time solution (3) for our problem in a basis of very effective high-localized functions, or nonlinear (in a sense of harmonic analysis) eigenmodes.
In addition to standard wavelet expansion on the whole real line which we used here, in calculation of the general Galerkin approximations, Melnikov function approach, etc. we need to use periodized wavelet expansion, i.e. wavelet expansion on finite interval [5]. Our approach works in such a case too.
Also, for the solution of perturbed system, we need to extend our approach to the important case of variable coefficients. For solving last problem we need to consider one more refinement equation for scaling function $\phi_2(x)$:

$$\phi_2(x) = \sum_{k=0}^{N-1} a_k^2 \phi_2(2x - k)$$

and corresponding wavelet expansion for variable coefficients:

$$b(t): \quad \sum_k B_k^j(b)\phi_2(2^j x - k),$$

where $B_k^j(b)$ are functionals supported in a small neighborhood of $2^{-j}k$ [7].

The solution of the first problem consists in periodizing. In this case we use expansion into periodized wavelets [5] defined by:

$$\phi^{per}_{-j,k}(x) = 2^{j/2}\sum_{Z}\phi(2^j x + 2^j\ell - k).$$

All these modifications lead to transformations of the coefficients of the reduced algebraic system only, but the general scheme described above remains the same. Extended versions and related results may be found in [11]–[18].

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