Invariant calculations in beam physics: dynamics on semi-direct products and CWT

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Abstract

We outline, according to Marsden's approach, the semi-direct product structure that allows to consider, from the general point of view, all kinematics groups such as Euclidean, Galilei, Poincare. Then we consider the proper invariant Lie-Poisson equations and present the manifestation of the semi-product structure of (kinematic) symmetry group on the dynamical level. After that, we consider the Lagrangian theory related to the semi-product structure and the explicit form of the variation principle and the corresponding (semi-direct) Euler-Poincare equations. All that provides the needful invariant background for CWT and the corresponding analytical technique that allows to consider covariant wavelet analysis. The proper orbit technique allows to construct different types of invariant wavelet bases which are very useful in applications to a number of the beam dynamics problems.

1 Introduction

In this paper, we briefly consider the dynamical consequences of some set of general invariant approaches based on investigation of a structure of the underlying hidden symmetry, which is the key property of any reasonable complex dynamical problem. Our main examples are located in the areas of accelerator/beam/plasma/quantum physics [1].

First of all, we are interested in covariance properties [2] regarding to relativity (kinematical) groups and our main instrument here is the so-called Continuous Wavelet Transform (CFT) as a method for investigation of dynamical properties [3].

In Section 2.1, we explain, according to very productive Marsden's approach [2], the semi-direct product structure, which allows us to consider all kinematical groups such as Euclidean, Galilei or Poincare from the general point of view in unified framework.

Then, in Section 2.2 we move to the dynamical consequences of such an algebraic description: we consider proper (invariant) Lie-Poisson equations and obtain the manifestation of semi-product structure of (kinematic) symmetry group on the dynamical level. So, as usually, the correct description of dynamics is a consequence of the correct understanding of the underlying symmetry of a concrete problem under

investigation. Strictly speaking, we consider right equations on representations of proper orbits generated by actions of hidden symmetries we like to take into account. In Section 2.3, we consider the technique for simplifications of dynamics related to semi-product structure by using reduction to the corresponding orbit structure. As a result, we have the simplified Lie-Poisson equations in the momentum map approach.
In Section 2.4 we move from the Lie-Poisson side to the Lagrangian one: we present the Lagrangian theory based on semi-product structure, the explicit form of the variation principle and the corresponding dual (semi-direct) Euler-Poicare equations [2].
Section 3 is devoted to Continuous Wavelet Transform [3] as a natural and high-power analytical invariant instrument for analysis on orbits and the corresponding analytical technique allows to consider covariant wavelet analysis, very important part of Nolinear (non-abelian) Local Harmonic Analysis [4] which is a very useful non-commutative and localized generalization of orthodox (abelian) Fourier Analysis.
In concluding Section 4, we consider the corresponding orbit technique for constructing different types of invariant wavelet bases in the important for us affine (Galilei) group with the semi-product structure. Such a technique, considered here, together with the related ones presented in the companion paper in this Volume [5], based on analysis of the underlying symplectic structure, was applied by authors to a number of physical problems of beam physics, accelerator physics, plasma physics and quantum physics. In this short exposition we constrain ourselves by ideological paradigm only. All details, constructions, and results can be found in [6]-[28].

2 Semi-direct Product Structures and Momentum Maps: From Lie-Poisson to Euler-Poincare

2.1 Semi-direct Product

Relativity groups such as Euclidean, Galilei or Poincare groups are the particular cases of semi-direct product construction, which is very useful and simple general construction in the group theory [2]. We may consider as a basic example the Euclidean group $SE(3) = SO(3) \bowtie \mathbf{R}^3$, the semi-direct product of rotations and translations. In general case we have $S = G \bowtie V$, where group G (Lie group or automorphisms group) acts on a vector space V and on its dual V^*. Let V be a vector space and G is the Lie group, which acts on the left by linear maps on V (G also acts on the left on its dual space V^*). The semi-direct product $S = G \bowtie V$ is the Cartesian product $S = G \times V$ with group multiplication

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2), \tag{1}$$

where the action of $g \in G$ on $v \in V$ is denoted as gv. Of course, we can consider the corresponding definitions both in case of the right actions and in case, when G is a group of automorphisms of the vector space V. As we shall explain below both cases, Lie groups and automorphisms groups, are important for us.

So, the Lie algebra of S is the semi-direct product Lie algebra, $s = \mathcal{G} \bowtie V$ with brackets

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1), \tag{2}$$

where the induced action of $\mathcal{G}$ by concatenation is denoted as $\xi_1 v_2$. Also we need expressions for adjoint and coadjoint actions for semi-direct products. Let $(g, v) \in S = G \times V$, $(\xi, u) \in s = \mathcal{G} \times V$, $(\mu, a) \in s^* = \mathcal{G}^* \times V^*$, $g\xi = Ad_g \xi$, $g\mu = Ad^*_{g^{-1}}\mu$, ga denotes the induced left action of g on a (the left action of G on V induces a left action on V^* — the inverse of the transpose of the action on V), $\rho_v : \mathcal{G} \to V$ is a linear map given by $\rho_v(\xi) = \xi v$, $\rho^*_v : V^* \to \mathcal{G}^*$ is its dual. Then these actions are given by simple concatenation:

$$\begin{aligned} (g, v)(\xi, u) &= (g\xi, gu - (g\xi)v), \\ (g, v)(\mu, a) &= (g\mu + \rho^*_v(ga), ga) \end{aligned} \tag{3}$$

Below we use the following notation: $\rho^*_v a = v \diamond a \in \mathcal{G}^*$ for $a \in V^*$, which is a bilinear operation in v and a. So, we have the coadjoint action:

$$(g, v)(\mu, a) = (g\mu + v \diamond (ga), ga). \tag{4}$$

Using concatenation notation for Lie algebra actions we have alternative definition of $v \diamond a \in \mathcal{G}^*$. For all $v \in V$, $a \in V^*$, $\eta \in \mathcal{G}$ we have

$$< \eta a, v >= - < v \diamond a, \eta > \tag{5}$$

2.2 The Lie-Poisson Equations and Semi-product Structure

Below we consider the manifestation of semi-product structure of symmetry group on dynamical level [2]. Let F, G be real valued functions on the dual space $\mathcal{G}^*$, $\mu \in \mathcal{G}^*$. Functional derivative of F at μ is the unique element $\delta F/\delta\mu \in \mathcal{G}$:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(\mu + \epsilon\delta\mu) - F(\mu)] =< \delta\mu, \frac{\delta F}{\delta\mu} > \tag{6}$$

for all $\delta\mu \in \mathcal{G}^*$, $<,>$ is pairing between $\mathcal{G}^*$ and $\mathcal{G}$.
Define the $(\pm)$ Lie-Poisson brackets by [2]

$$\{F, G\}_\pm(\mu) = \pm < \mu, [\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}] > \tag{7}$$

The Lie-Poisson equations, determined by

$$\dot{F} = \{F, H\} \tag{8}$$

read intrinsically

$$\dot{\mu} = \mp ad^*_{\partial H/\partial\mu}\mu. \tag{9}$$

For the left representation of G on V $\pm$ Lie-Poisson bracket of two functions $f, k : s^* \to \mathbf{R}$ is given by

$$\{f,k\}_{\pm}(\mu,a) = \pm < \mu, [\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}] > \pm < a, \frac{\delta f}{\delta \mu}\frac{\delta k}{\delta a} - \frac{\delta k}{\delta \mu}\frac{\delta f}{\delta a} >, \tag{10}$$

where $\delta f/\delta \mu \in \mathcal{G}$, $\delta f/\delta a \in V$ are the functional derivatives of f (6). The Hamiltonian vector field of $h : s^* \in \mathbf{R}$ has the expression

$$X_h(\mu,a) = \mp(ad^*_{\delta h/\delta\mu}\mu - \frac{\delta h}{\delta a} \diamond a, -\frac{\delta h}{\delta \mu} a). \tag{11}$$

Thus, Hamiltonian equations on the dual of a semi-direct product are [2]:

$$\begin{aligned} \dot{\mu} &= \mp ad^*_{\delta h/\delta\mu}\mu \pm \frac{\delta h}{\delta a} \diamond a \\ \dot{a} &= \pm \frac{\delta h}{\delta \mu} a \end{aligned} \tag{12}$$

So, we can see the explicit difference between Poisson brackets (7) and (10) and the equations of motion (9) and (12), which come from the semi-product structure.

2.3 Reduction of Dynamics on Semi-product

There is a technique for reducing dynamics that is associated with the geometry of semi-direct product reduction theorem[2]. Let us have a Hamiltonian on T^*G that is invariant under the isotropy G_{a_0} for $a_0 \in V^*$. The semi-direct product reduction theorem states that reduction of T^*G by G_{a_0} gives reduced spaces that are simplectically diffeomorphic to coadjoint orbits in the dual of the Lie algebra of the semi-direct product $(\mathcal{G} \bowtie V)^*$. If one reduces the semi-direct group product $S = G \bowtie V$ in two stages, first by V and then by G one recovers this semi-direct product reduction theorem. Thus, let $S = G \bowtie V$, choose $\sigma = (\mu, a) \in \mathcal{G}^* \times V^*$ and reduce T^*S by the action of S at σ giving the coadjoint orbit $\mathcal{O}_\sigma$ through $\sigma \in S^*$. There is a symplectic diffeomorphism between $\mathcal{O}_\sigma$ and the reduced space obtained be reducing T^*G by the subgroup G_a (the isotropy of G for its action on V^* at the point $a \in V^*$) at the point $\mu|\mathcal{G}_a$, where $\mathcal{G}_a$ is the Lie algebra of G_a.
Then we have the following procedure.

1. We start with a Hamiltonian H_{a_0} on T^*G that depends parametrically on a variable $a_0 \in V^*$.

2. The Hamiltonian regarded as a map: $T^*G \times V^* \to \mathbf{R}$ is assumed to be invariant on T^*G under the action of G on $T^*G \times V^*$.

3. The condition 2 is equivalent to the invariance of the function H defined on $T^*S = T^*G \times V \times V^*$ extended to be constant in the variable V under the action of the semi-direct product.

4. By the semi-direct product reduction theorem, the dynamics of H_{a_0} reduced by G_{a_0}, the isotropy group of a_0, is simplectically equivalent to Lie-Poisson dynamics on $s^* = \mathcal{G}^* \times V^*$.

5. This Lie-Poisson dynamics is given by equations (12) for the function $h(\mu, a) = H(\alpha_g, g^{-1}a)$, where $\mu = g^{-1}\alpha_g$.

2.4 Lagrangian Theory, the Euler-Poincare Equations, Variational Approach on Semi-product

Now we consider according to [2] Lagrangian side of a theory. This approach is based on variational principles with symmetry and is not dependent on Hamiltonian formulation, although it is well-known that this purely Lagrangian formulation is equivalent to the Hamiltonian formulation on duals of semi-direct product (the corresponding Legendre transformation is a diffeomorphism).
We consider the case of the left representation and the left invariant Lagrangians (ℓ and L), which depend in additional on another parameter $a \in V^*$ (dynamical parameter), where V is representation space for the Lie group G and L has an invariance property related to both arguments. It should be noted that the resulting equations of motion, the Euler-Poincare equations, are not the Euler-Poincare equations for the semi-direct product Lie algebra $\mathcal{G} \bowtie V^*$ or $\mathcal{G} \bowtie V$.
So, we have the following:

1. There is a left presentation of Lie group G on the vector space V and G acts in the natural way on the left on $TG \times V^* : h(v_g, a) = (hv_g, ha)$.

2. The function $L : TG \times V^* \in \mathbf{R}$ is the left G-invariant.

3. Let $a_0 \in V^*$, Lagrangian $L_{a_0} : TG \to \mathbf{R}$, $L_{a_0}(v_g) = L(v_g, a_0)$. L_{a_0} is left invariant under the lift to TG of the left action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.

4. Left G-invariance of L permits us to define
$$\ell : \mathcal{G} \times V^* \to \mathbf{R} \tag{13}$$
by
$$\ell(g^{-1}v_g, g^{-1}a_0) = L(v_g, a_0). \tag{14}$$
This relation defines for any $\ell : \mathcal{G} \times V^* \to \mathbf{R}$ the left G-invariant function $L : TG \times V^* \to \mathbf{R}$.

5. For a curve $g(t) \in G$ let be
$$\xi(t) := g(t)^{-1}\dot{g}(t) \tag{15}$$
and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients
$$\dot{a}(t) = -\xi(t)a(t), \tag{16}$$
with initial condition $a(0) = a_0$. The solution can be written as $a(t) = g(t)^{-1}a_0$.

Then we have four equivalent descriptions of the corresponding dynamics:

1. If a_0 is fixed then Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t)\mathrm{d}t = 0 \tag{17}$$

holds for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

2. $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.

3. The constrained variational principle

$$\delta \int_{t_1}^{t_2} \ell(\xi(t), a(t))\mathrm{d}t = 0 \tag{18}$$

holds on $\mathcal{G} \times V^*$, using variations of ξ and a of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$, $\delta a = -\eta a$, where $\eta(t) \in \mathcal{G}$ vanishes at the endpoints.

4. The Euler-Poincare equations hold on $\mathcal{G} \times V^*$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta \xi} = ad^*_\xi \frac{\delta \ell}{\delta \xi} + \frac{\delta \ell}{\delta a} \diamond a \tag{19}$$

So, we may apply our wavelet methods either on the level of variational formulation (17) or on the level of Euler-Poincare equations (19).

3 Continuous Wavelet Transform

At this point, we need to take into account all features of the Hamiltonian or Lagrangian structures related with systems covered by (12) or (19). Therefore, we need to consider generalized (covariant) wavelets. It allows us to consider the corresponding invariant representations instead of standard and non-covariant ones, e.g., compactly supported wavelet decompositions [3].

In Nonlinear Local Harmonic Analysis the following three concepts are used now: 1). a square integrable representation U of a group G, 2). coherent states (CS) over G, 3). the wavelet transform associated to U. We consider now their unification [3], [4], [29], [30].

Let G be a locally compact group and U_a strongly continuous, irreducible, unitary representation of G on Hilbert space $\mathcal{H}$. Let H be a closed subgroup of G, $X = G/H$ with (quasi) invariant measure ν and $\sigma : X = G/H \to G$ is a Borel section in a principal bundle $G \to G/H$. Then we say that U is square integrable $mod(H, \sigma)$ if there exists a non-zero vector $\eta \in \mathcal{H}$ such that

$$0 < \int_X | < U(\sigma(x))\eta|\Phi > |^2 \mathrm{d}\nu(x) =< \Phi|A_\sigma\Phi > < \infty, \quad \forall \Phi \in \mathcal{H} \tag{20}$$

Given such a vector $\eta \in \mathcal{H}$ called admissible for (U, σ) we define the family of (covariant) coherent states or wavelets, indexed by points $x \in X$, as the orbit of η under G, though the representation U and the section σ [29], [30]:

$$S_\sigma = \eta_{\sigma(x)} = U(\sigma(x))\eta | x \in X \tag{21}$$

So, coherent states or wavelets are simply the elements of the orbit under U of a fixed vector η in representation space. We have the following fundamental properties:

1. Overcompleteness:
   The set S_σ is total in $\mathcal{H} : (S_\sigma)^\perp = 0$

2. Resolution property:
   the square integrability condition (20) may be represented as a resolution relation:

$$\int_X |\eta_\sigma(x) >< \eta_{\sigma(x)}|\mathrm{d}\nu(x) = A_\sigma, \tag{22}$$

   where A_σ is a bounded, positive operator with a densely defined inverse. Define the linear map

$$W_\eta : \mathcal{H} \to L^2(X, \mathrm{d}\nu), (W_\eta\Phi)(x) =< \eta_{\sigma(x)}|\Phi > \tag{23}$$

   Then the range H_η of W_η is complete with respect to the scalar product $< \Phi|\Psi >_\eta =< \Phi|W_\eta A_\sigma^{-1} W_\eta^{-1}\Psi >$ and W_η is unitary operator from $\mathcal{H}$ onto $\mathcal{H}_\eta$. W_η is Continuous Wavelet Transform (CWT).

3. Reproducing kernel
   The orthogonal projection from $L^2(X, \mathrm{d}\nu)$ onto $\mathcal{H}_\eta$ is an integral operator K_σ and H_η is a reproducing kernel Hilbert space of functions:

$$\Phi(x) = \int_X K_\sigma(x, y)\Phi(y)\mathrm{d}\nu(y), \quad \forall\Phi \in \mathcal{H}_\eta. \tag{24}$$

   The kernel is given explicitly by $K_\sigma(x, y) =< \eta_{\sigma(x)}|A_\sigma^{-1}\eta_{\sigma(y)} >$, if $\eta_{\sigma(y)} \in D(A_\sigma^{-1})$, $\forall y \in X$. So, the function $\Phi \in L^2(X, \mathrm{d}\nu)$ is a wavelet transform (WT) iff it satisfies this reproducing relation.

4. Reconstruction formula.
   The WT W_η may be inverted on its range by the adjoint operator, $W_\eta^{-1} = W_\eta^*$ on $\mathcal{H}_\eta$ to obtain for $\eta_{\sigma(x)} \in D(A_\sigma^{-1})$, $\forall x \in X$

$$W_\eta^{-1}\Phi = \int_X \Phi(x)A_\sigma^{-1}\eta_{\sigma(x)}\mathrm{d}\nu(x), \quad \Phi \in \mathcal{H}_\eta. \tag{25}$$

   This is inverse WT.

If A_σ^{-1} is bounded then S_σ is called a frame, if $A_\sigma = \lambda I$ then S_σ is called a tight frame. This two cases are generalization of a simple case, when S_σ is an (ortho)basis. The most simple cases of this construction are:
1. $H = \{e\}$. This is the standard construction of WT over a locally compact group. It should be noted that the square integrability of U is equivalent to U belonging to the discrete series. The most simple example is related to the affine $(ax + b)$ group and yields the usual one-dimensional wavelet analysis

$$[\pi(b, a)f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x - b}{a}\right). \tag{26}$$

For $G = SIM(2) = \mathbf{R}^2 \bowtie (\mathbf{R}_*^+ \times SO(2))$, the similitude group of the plane, we have the corresponding two-dimensional wavelets.
2. $H = H_\eta$, the isotropy (up to a phase) subgroup of η: this is the case of the Gilmore-Perelomov CS. Some cases of group G are:
a). Semisimple groups, such as SU(N), SU(N|M), SU(p,q), Sp(N,$\mathbf{R}$).
b). the Weyl-Heisenberg group G_{WH} which leads to the Gabor functions, i.e. canonical (oscillator)coherent states associated with windowed Fourier transform or Gabor transform [4], [5]:

$$[\pi(q,p,\varphi)f](x) = \exp(i\mu(\varphi - p(x-q))f(x-q) \qquad (27)$$

In this case H is the center of G_{WH}. In both cases time-frequency plane corresponds to the phase space of group representation.
c). The similitude group SIM(n) of $\mathbf{R}^n (n \geq 3)$: for $H = SO(n-1)$ we have the axisymmetric n-dimensional wavelets.
d). Also we have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl–Heisenberg group [30].
e). Relativity groups. In a nonrelativistic setup, the natural kinematical group is the (extended) Galilei group. Also we may adds independent space and time dilations and obtain affine Galilei group. If we restrict the dilations by the relation $a_0 = a^2$, where a_0, a are the time and space dilation we obtain the Galilei-Schrödinger group, invariance group of both Schrödinger and heat equations. We consider these examples in the next section. In the same way we may consider as kinematical group the Poincare group. When $a_0 = a$ we have affine Poincare or Weyl-Poincare group. Some useful generalization of that affinization construction we consider for the case of hidden metaplectic structure in companion paper [5].
But the usual representation is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in our case) restores square – integrability: $G \longrightarrow$ homogeneous space. Also, we can consider much more general approach which allows to describe generalized wavelets corresponding to more general groups and representations [29], [30].
Our final goal is the application of these results to Hamiltonian/Lie-Poissonian background and as a consequence we need to take into account symplectic/Poissonian nature of our dynamical problems [5]. So, the symplectic and wavelet structures must be consistent (like we have in the symplectic or Lie-Poisson integrator theory). We hope to use the point of view of geometric quantization theory (orbit method) instead of orthodox harmonic analysis, so it seems that we can consider the points (a)–(e) above in the unified framework.

4 Invariant Bases for Solutions

We consider an important particular case of affine relativity group (relativity group combined with dilations) — affine Galilei group in n-dimensions. So, we have combination of Galilei group with independent space and time dilations: $G_{aff} = G_m \bowtie D_2$, where $D_2 = (\mathbf{R}_*^+)^2 \simeq \mathbf{R}^2$, G_m is extended Galilei group corresponding to mass parameter $m > 0$ (G_{aff} is noncentral extension of $G \bowtie D_2$ by $\mathbf{R}$, where G is usual Galilei group). Generic element of G_{aff} is $g = (\Phi, b_0, b; v; R, a_0, a)$, where $\Phi \in \mathbf{R}$ is the extension parameter in G_m, $b_0 \in \mathbf{R}$, $b \in \mathbf{R}^n$ are the time and space translations, $v \in \mathbf{R}^n$ is the boost parameter, $R \in SO(n)$ is a rotation and $a_0, a \in \mathbf{R}_*^+$ are time and space dilations. The actions of g on space-time is then $x \mapsto aRx + a_0vt + b$, $t \mapsto a_0t + b_0$, where $x = (x_1, x_2, ..., x_n)$. The group law is [4], [29], [30]:

$$\begin{aligned} gg' = \quad & (\Phi + \frac{a^2}{a_0}\Phi' + avRb' + \frac{1}{2}a_0v^2b_0', b_0 + a_0b_0', b + aRb' + a_0vb_0'; \\ & v + \frac{a}{a_0}Rv', RR'; a_aa_0', aa') \end{aligned} \tag{28}$$

It should be noted that D_2 acts nontrivially on G_m. Space-time wavelets associated to G_{aff} corresponds to unitary irreducible representation of spin zero. It may be obtained via orbit method. The Hilbert space is $\mathcal{H} = L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega)$, $k = (k_1, ..., k_n)$, where $\mathbf{R}^n \times \mathbf{R}$ may be identified with usual Minkowski space and we have for representation:

$$(U(g)\Psi)(k, \omega) = \sqrt{a_0a^n}\mathrm{exp}i(m\Phi + kb - \omega b_0)\Psi(k', \omega'), \tag{29}$$

with $k' = aR^{-1}(k + mv)$, $\omega' = a_0(\omega - kv - \frac{1}{2}mv^2)$, $m' = (a^2/a_0)m$. Mass m is a coordinate in the dual of the Lie algebra and these relations are a part of coadjoint action of G_{aff}. This representation is unitary and irreducible but not square integrable. So, we need to consider reduction to the corresponding quotients $X = G/H$. We consider the case in which H={phase changes Φ and space dilations a}. Then the space $X = G/H$ is parametrized by points $\bar{x} = (b_0, b; v; R; a_0)$. There is a dense set of vectors $\eta \in \mathcal{H}$ admissible mod(H, σ_β), where σ_β is the corresponding section. We have a two-parameter family of functions β(dilations): $\beta(\bar{x}) = (\mu_0 + \lambda_)a_0)^{1/2}$, $\lambda_0, \mu_0 \in \mathbf{R}$. Then any admissible vector η generates a tight frame of Galilean wavelets [4], [29]:

$$\eta_{\beta(\bar{x})}(k, \omega) = \sqrt{a_0(\mu_0 + \lambda_0a_0)^{n/2}}\mathrm{e}^{i(kb - \omega b_0)}\eta(k', \omega'), \tag{30}$$

with $k' = (\mu_0 + \lambda_0a)^{1/2}R^{-1}(k + mv)$, $\omega' = a_0(\omega - kv - mv^2/2)$. The simplest examples of admissible vectors (corresponding to usual Galilei case) are Gaussian vector: $\eta(k) \sim \exp(-k^2/2mu)$ and binomial vector: $\eta(k) \sim (1 + k^2/2mu)^{-\alpha/2}$, $\alpha > 1/2$, where u is a kind of internal energy. When we impose the relation $a_0 = a^2$ then we have the restriction to the Galilei-Schrödinger group $G_s = G_m \bowtie D_s$, where D_s is the one-dimensional subgroup of D_2. G_s is a natural invariance group of both the Schrödinger equation and the heat equation. The restriction to G_s of the representation (29) splits into the direct sum of two irreducible ones $U = U_+ \oplus U_-$ corresponding to the decomposition $L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega) = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\begin{aligned} \mathcal{H}_\pm &= L^2(D_\pm, \mathrm{d}k\mathrm{d}\omega \\ &= \{\psi \in L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega), \quad \psi(k, \omega) = 0 \quad for \quad \omega + k^2/2m = 0\} \end{aligned} \tag{31}$$

These two subspaces are the analogues of usual Hardy spaces on $\mathbf{R}$, i.e. the subspaces of (anti)progressive wavelets (see also [5]). The two representation $U_\pm$ are square

integrable modulo the center. There is a dense set of admissible vectors η, and each of them generates a set of CS of Gilmore-Perelomov type. Typical wavelets of this kind are [4], [29]:
the Schrödinger-Marr wavelet:

$$\eta(x,t) = (i\partial_t + \frac{\triangle}{2m})\mathrm{e}^{-(x^2+t^2)/2} \tag{32}$$

the Schrödinger-Cauchy wavelet:

$$\psi(x,t) = (i\partial_t + \frac{\triangle}{2m})\frac{1}{(t+i)\prod_{j=1}^{n}(x_j+i)} \tag{33}$$

So, in the same way we can construct invariant bases with explicit manifestation of the underlying symmetry to solve Hamiltonian (12) or Lagrangian (19) type of background equations.

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