Beam-beam interaction: from localization to non-gaussian spectrum

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Abstract

We consider very important in the high-energy beam physics and in plasma physics numerical-analytical modeling for the process of strong-strong beam-beam interactions beyond standard linearized/perturbative methods such as soft gaussian approximation, or Fast Multipole Method (FMM), or related Hybrid FMM, etc. In our approach, the full spectrum of beam-beam interaction consists of discrete coherent modes, discovered before, and the zoo of stochastic incoherent oscillations, appearing as a result of the complex nonlinear inter-mode evolution in the full tower of hidden internal fundamental (eigen)modes or some sort of interference between orbits of the representation of the hidden generic symmetry of the underlying functional space. We consider the proper multiresolution/multiscale fast convergent decomposition in the bases of high-localized exact nonlinear modes represented by wavelets or wavelet packets as the best tool, allowing to describe the most important in many areas of high-energy physics non-gaussian effects leading to non-trivial dynamical effects, which are very important in the modern accelerator and plasma physics. The constructed solutions represent the full multiscale spectrum in all internal hidden scales, starting from coarse-grained approximation to finest one. The underlying variational method provides, in principle, the possibility for the algebraical control of spectral data.

1 Introduction

We consider the first steps of novel analysis of beam-beam interactions in some collective model approach. It is well-known that neither direct Particle-in-Cell (PIC) modeling nor soft-gaussian approximation provide the reasonable resolution of computing time/noise problems and understanding of the underlying complex nonlinear dynamics [1], [2]. Recent analysis based both on numerical simulation and modeling, demonstrates that the presence of coherent modes inside the spectrum leads to oscillations and the growth of beam transverse size and deformations of beam shape and, as a result, this leads to strong limitations for the operation of Large Hadron Collider (LHC) and other power machines. Additional problems appear as a result of the influence of the continuum spectrum of incoherent oscillations on beam-beam interactions. The strong-strong collisions of two beams also lead to the variation of transverse size. According to analysis in [2], it is reasonable to find nonperturbative and/or non-gaussian solutions at least in the important generic cases. Our approach based on Multiresolution Decomposition in the framework of general Local Nonlinear Harmonic Analysis (wavelet analysis in the simple case) technique is in some sense the direct generalization of Fast Multipole Method (FMM) and related approaches, like Hybrid FMM (HFMM). After set-up in Section 2, based on Vlasov-like models (according to exposition in [2], [3]), in Section 3, we consider the variational–multiresolution approach [4]-[15] in framework of powerful technique based on the operator representation by Fast Wavelet Transform (FWT)[16], [17]. As a result we represent the complex dynamics of beam-beam interaction via multiscale fast convergent decomposition in the bases of high-localized exact nonlinear non-gaussian eigenmodes represented by wavelets or wavelet packets functions. The constructed solutions represent the full multiscale spectrum of the underlying dynamics in all internal hidden scales from slow modes to fast oscillating ones, from coarse graining approximation to finest one. The underlying variational method provides the algebraical control of spectrum data, allowing to organize, in principle, some sort of the control for beam-beam interaction.

2 Vlasov model for beam-beam interactions

Vlasov-like equations describing evolution of the phase space distributions $\psi^j = \psi^j(x, p_x, \theta)$ $(j = 1, 2)$ for each beam are [2]:

$$\frac{\partial \psi^j}{\partial \theta} = -q_x p_x \frac{\partial \psi^j}{\partial x} + \Big(q_x x + \delta_p(\theta) 4\pi \xi_x p.v. \int_{-\infty}^{\infty} \frac{\rho^*(x', \theta)}{x - x'} \mathrm{d}x'\Big) \frac{\partial \psi^j}{\partial p_x}$$

where

$$\rho^*(x, \theta) = \int_{-\infty}^{\infty} \psi^*(x, p_x, \theta) \mathrm{d}p_x \tag{1}$$

and ψ^* is the density of the opposite beam, q_x is unperturbed fractional tune, ξ_x is horizontal beam-beam parameter, N is a number of particles, x, p_x are normalized variables.

This model describes horizontal oscillations of flat beams with one bunch per beam, one interaction point, equal energy, population and optics for both beams.

3 FWT based variational approach

One of the key points of wavelet analysis approach demonstrates that for a large class of operators wavelets are good approximation for true eigenvectors and the corresponding matrices are almost diagonal. FWT [17] provides the maximum sparse form of general classes of (pseudodifferential) operators.

Definitely, it is true also in case of operators involved in our Vlasov-like system of equations (1). We have both differential and integral operators inside.

So, let us denote our (integral/differential) operator from equations (1) as

$$T: \quad (L^2(R^n) \to L^2(R^n)) \tag{2}$$

and its kernel as K. So, we have the following representation for the matrix elements:

$$< Tf, g >= \int\int K(x, y) f(y) g(x) \mathrm{d}x \mathrm{d}y \tag{3}$$

In case when f and g are wavelets generated by action of underlying affine group of translations and dilations

$$\varphi_{j,k} = 2^{j/2} \varphi(2^j x - k) \tag{4}$$

the representation (3) provides the standard representation for operator T.

Let us consider multiresolution representation

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots \tag{5}$$

The basis in each V_j is $\varphi_{j,k}(x)$, where indices k, j represent translations and scaling, respectively. Let

$$P_j : L^2(R^n) \to V_j \qquad (j \in Z) \tag{6}$$

be projection operators on the subspace V_j corresponding to level j of resolution:

$$(P_j f)(x) = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x). \tag{7}$$

Let

$$Q_j = P_{j-1} - P_j \tag{8}$$

be the projection operator on the subspace W_j defined by relation

$$V_{j-1} = V_j \oplus W_j, \tag{9}$$

then we have the following "microscopic or telescopic" representation of operator T which takes into account contributions from each level of resolution from different scales starting with the coarsest and ending to the finest scales [17]:

$$T = \sum_{j \in Z} (Q_j T Q_j + Q_j T P_j + P_j T Q_j). \tag{10}$$

We remember that this is a result of presence of internal hidden symmetry in underlying functional space, namely affine group, which generates all such constructions.

The non-standard form of operator representation [17] is a representation of operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j \in Z}$, acting on the subspaces V_j and W_j:

$$A_j : W_j \to W_j, B_j : V_j \to W_j, \Gamma_j : W_j \to V_j, \tag{11}$$

where operators $\{A_j, B_j, \Gamma_j\}_{j \in Z}$ are defined as

$$A_j = Q_j T Q_j, \quad B_j = Q_j T P_j, \quad \Gamma_j = P_j T Q_j. \tag{12}$$

The operator T admits a recursive definition via

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix}, \tag{13}$$

where

$$T_j = P_j T P_j \quad \text{and} \quad T_j \quad \text{acts on} \quad V_j : V_j \to V_j. \tag{14}$$

It should be noted that operator A_j describes interaction on the scale j independently from other scales, operators B_j, Γ_j describe interaction between the scale j and all coarser scales, the operator T_j is an "averaged" version of T_{j-1}.

We may create such a non-standard representation for various classes of operators including Calderon-Zygmund or pseudodifferential.

But, both in the case of differential operator and in other cases all we need is only to solve the system of linear algebraical equations. It is a big advantage of this power FWT approach.

The action of integral operator involved into the equations (1) we may consider as a Hilbert transform

$$(H\rho^*)(x) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\rho^*(x', \theta)}{x' - x} \mathrm{d}x' \tag{15}$$

The representation of H on V_0 is defined by the coefficients

$$r_\ell = \int \varphi(x - \ell)(H\varphi)(x)\mathrm{d}x, \quad \ell \in Z, \tag{16}$$

which, according to FWT technique, defines also all other coefficients of the nonstandard representation.

So, we have the following triple representation for $H = \{A_j, B_j, \Gamma_j\}_{j \in Z}$ with the corresponding matrix elements $a_{i-\ell}, b_{i-\ell}, c_{i-\ell}$ which can be computed from coefficients r_ℓ only:

$$\begin{aligned} a_i &= \sum_{k,k'=0}^{L-1} g_k g_{k'} r_{2i+k-k'} \\ b_i &= \sum_{k,k'=0}^{L-1} g_k h_{k'} r_{2i+k-k'} \\ c_i &= \sum_{k,k'=0}^{L-1} h_k g_{k'} r_{2i+k-k'} \end{aligned} \tag{17}$$

The coefficients r_ℓ (16) can be obtained from

$$r_\ell = r_{2\ell} + \sum_{k=1}^{L/2} d_{2k-1}(r_{2\ell-2k+1} + r_{2\ell+2k-1}) \tag{18}$$

where d_n are the so called autocorrelation coefficients of the corresponding quadratic mirror filter $\{h_k\}_{k=0}^{L-1}$:

$$\begin{aligned} d_n &= 2\textstyle\sum_{i=0}^{L-1-n} h_i h_{i+n}, && n = 1, \ldots, L-1, \\ d_{2k} &= 0, && k = 1, \ldots, L/2 - 1, \\ g_k &= (-1)^k h_{L-k-1}, && k = 0, \ldots, L-1, \end{aligned} \tag{19}$$

which parametrizes the generic refinement equation

$$\varphi(x) = \sqrt{2} \sum_{k=0}^{L-1} h_k \varphi(2x - k). \tag{20}$$

This equation really generates all wavelet zoo. It is useful to add to the system (18) the following asymptotic condition $r_\ell = -1/\pi\ell + O(\ell^{-2M})$, which simplifies the solution procedure.

Then, finally, we have the following action of operator T_j on sufficiently smooth function f:

$$(T_j f)(x) = \sum_{k \in Z} \left(2^{-j} \sum_\ell r_\ell f_{j,k-\ell} \right) \varphi_{j,k}(x) \tag{21}$$

in the wavelet basis $\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x - k)$, where

$$f_{j,k-1} = 2^{-j/2}\int f(x)\varphi(2^{-j}x - k + \ell)\mathrm{d}x \tag{22}$$

are wavelet coefficients.

So, as a principal result, we have simple linear parametrization of matrix representation for our key operator (15) in the wavelet bases and as a byproduct we may compute the action of this operator on arbitrary vector in proper functional space. The similar approach can be applied to other operators involved in (1) [16], [17].

After all that we are ready to apply our variational approach from [4]-[15]. Let L be an arbitrary (non) linear (differential/integral) operator corresponds to the system (1) with matrix dimension d, which acts on some set of functions

$$\begin{aligned} \Psi &\equiv \Psi(\theta,x,p_x) = \Big(\Psi^1(\theta,x,p_x),\dots,\Psi^d(\theta,x,p_x)\Big), \quad \theta,x,p_x \in \Omega \subset \mathbf{R}^3, \\ L\Psi &\equiv L(Q,\theta,x,p_x)\Psi(\theta,x,p_x) = 0, \end{aligned} \tag{23}$$

where

$$Q \equiv Q_{d_1,d_2,d_3}(\theta,x,p_x,\partial/\partial\theta,\partial/\partial x,\partial/\partial p_x,\int \mathrm{d}x\mathrm{d}p_x).$$

Let us consider now the N mode approximation for solution as the following ansatz (in the same way we may consider different ansatzes) [15]:

$$\Psi^N(\theta,x,p_x) = \sum_{r,s,k=1}^{N} a_{rsk}A_r \otimes B_s \otimes C_k(\theta,x,p_x) \tag{24}$$

We shall determine the coefficients of expansion from the following conditions (various related variational approaches are considered in [4]-[15]):

$$\ell^N_{k\ell m} \equiv \int (L\Psi^N)A_k(\theta)B_\ell(x)C_m(p_x)\mathrm{d}\theta\mathrm{d}x\mathrm{d}p_x = 0. \tag{25}$$

So, we have exactly dN^3 algebraical equations for dN^3 unknowns a_{rsk}. The solution is parametrized by solutions of two set of reduced algebraical problems, one is linear or nonlinear (depends on the structure of operator L) and the rest are some linear problems related to computation of coefficients of algebraic equations. These coefficients can be found by some multiresolution machinery by using, e.g., compactly supported wavelet basis functions for expansions (24). We may consider also different types of wavelets including general wavelet packets. The constructed solution has the following multiscale/multiresolution decomposition via nonlinear high-localized non-gaussian eigenmodes

$$\psi(\theta,x,p_x) = \sum_{(i,j,k)\in Z^3} a_{ijk}A^i(\theta)B^j(x)C^k(p_x), \tag{26}$$

$$A^i(\theta) = A^{i,slow}_N(\theta) + \sum_{r\geq N} A^i_r(\omega_r\theta),\ \omega_r \sim 2^r$$

$$B^j(x) = B^{j,slow}_M(x) + \sum_{l\geq M} B^j_l(k^1_l x),\ k^1_l \sim 2^l$$

$$C^s(p_x) = C^{s,slow}_L(p_x) + \sum_{m\geq L} C^s_m(k^2_m p_x),\ k^2_m \sim 2^m$$

which corresponds to the full multiresolution expansion in all underlying time/space scales starting from coarse-grained approximation.

Formulas (26) give us an expansion into the slow part $f^{slow}_{N,M,L}$ and fast oscillating parts for arbitrary N, M, L. So, we may move from coarse scales of resolution to the finest one to obtain more detailed information about our dynamical process. The first terms in the RHS of formulas (26) correspond on the global level of function space decomposition to resolution space and the second ones to detail space. The using of wavelet basis with high-localized properties provides the fast convergence of constructed decomposition (26).

In contrast with different approaches, formulas (26) do not use perturbation technique or linearization procedures and represent the non-gaussian part of spectrum which is the most complicated part of non-trivial dynamics of beam-beam interaction.

Numerical calculations are based on compactly supported wavelets and related wavelet families [18]. On Figure 1 we present fundamental localized non-gaussian eigenmodes used for modeling by representation (26). Figures 2,3 demonstrate different types of complex dynamics of beam-beam interactions: weak and strong regimes.

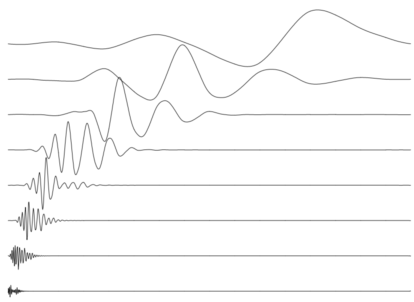

Figure 1: Non-Linear/Non-Gaussian Eigenmodes.

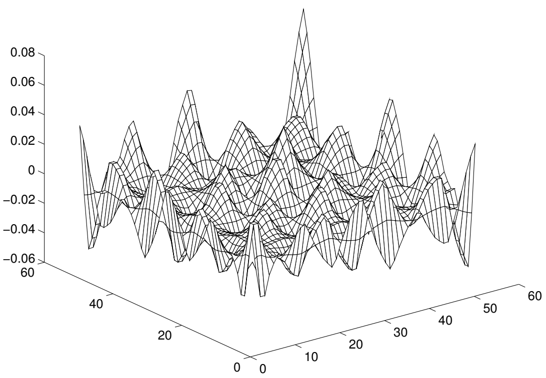

Figure 2: Scattering Dynamics: Weak Regime.

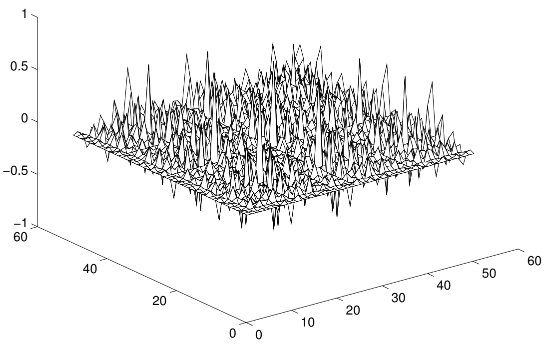

Figure 3: Scattering Dynamics: Strong Regime.

Definitely, the more complicated situation demands to take into account a needful number of hidden scales and, of course, we have no chances for adequate modeling by using coarse-grained gaussian approximations only. It should be noted that algebraic equations (25), the so-called Generalized Dispersion Relation [4]-[15], open the way for possible pure algebraic control of complicated scattering dynamics. Details will be considered elsewhere.

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